The anabelian geometry of configuration spaces of hyperbolic curves in positive characteristic

Yuichiro Hoshi, Koichiro Sawada and Shota Tsujimura

Abstract

In the present paper, we prove the anabelian Grothendieck conjecture for the tame fundamental groups of the configuration spaces associated to hyperbolic curves over [the perfections of] finitely generated fields of positive characteristic. The main theorem of the present paper generalizes the classical anabelian results for hyperbolic curves in positive characteristic established by A. Tamagawa, S. Mochizuki, and J. Stix. The main theorem of the present paper may also be regarded as the first anabelian Grothendieck conjecture-type result for algebraic varieties in positive characteristic of higher dimension [i.e., of dimension greater than one]. In the process of the proof of the main theorem, we prove a certain exactness of homotopy sequences for the tame fundamental groups with respect to suitable morphisms between normal varieties. Moreover, we also introduce the notion of a generalized fiber subgroup of the tame fundamental group of the configuration space associated to a hyperbolic curve in arbitrary characteristic and establish a "group-theoretic algorithm" that reconstructs, from the tame fundamental group of the configuration space, the generalized fiber subgroups.

Contents

Introduction

In the present paper, subsequent to the recent work of the third author, from the viewpoint of the compatibility/rigidity/synchronization of group-theoretic cyclotomes, we study the anabelian Grothendieck conjecture for the configuration spaces associated to hyperbolic curves over [the perfections of] finitely generated fields of positive characteristic.

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Let us first recall famous classical anabelian results for hyperbolic curves in positive characteristic established by A. Tamagawa, S. Mochizuki, and J. Stix. For each $\square \in \{\dagger, \ddagger\}$, let

- $^{\square}k$ be a field,
- $^{\square}\overline{k}$ a separable closure of $^{\square}k$, and
- $^{\square}X^{+} = (^{\square}X, {}^{\square}D)$ a hyperbolic curve over $^{\square}k$ [cf. Definition 2.1].

For each $\square \in \{\dagger, \ddagger\}$, write

- $^{\square}X^{+}_{^{\square}\overline{k}} \overset{\mathrm{def}}{=} (^{\square}X \times_{^{\square}k} {}^{\square}\overline{k}, {}^{\square}D \times_{^{\square}k} {}^{\square}\overline{k})$ for the hyperbolic curve over $^{\square}\overline{k}$ obtained by forming the base-change of $^{\square}X^{+}$ to $^{\square}\overline{k}$,
- $^{\square}U \overset{\mathrm{def}}{=} {}^{\square}X \setminus {}^{\square}D \subseteq {}^{\square}X$ for the open subscheme of $^{\square}X$ obtained by forming the complement of $^{\square}D$ in $^{\square}X$,
- $G_{^{\square}k} \overset{\mathrm{def}}{=} \mathrm{Gal}(^{\square}\overline{k}/{}^{\square}k)$ for the absolute Galois group of the field $^{\square}k$ determined by the separable closure $^{\square}\overline{k}$,
- $^{\square}\Pi \overset{\mathrm{def}}{=} \pi_1^{\mathrm{tame}}(^{\square}X^{+})$ for the tame fundamental group of $^{\square}X^{+} = (^{\square}X, {}^{\square}D)$, relative to a suitable choice of basepoint, and
- $^{\square}\Delta \overset{\mathrm{def}}{=} \pi_1^{\mathrm{tame}}(^{\square}X^{+}_{^{\square}\overline{k}})$ for the tame fundamental group of $^{\square}X^{+}_{^{\square}\overline{k}} = (^{\square}X \times_{^{\square}k} {}^{\square}\overline{k}, {}^{\square}D \times_{^{\square}k} {}^{\square}\overline{k})$, relative to a suitable choice of basepoint.

Thus, for each $\square \in \{\dagger, \ddagger\}$, the natural morphisms $^{\square}X_{^{\square}\overline{k}} \to {}^{\square}X \to \mathrm{Spec}(^{\square}k)$ determine an exact sequence of topological groups

$$1 \longrightarrow {}^{\square}\Delta \longrightarrow {}^{\square}\Pi \longrightarrow G_{^{\square}k} \longrightarrow 1.$$

Now let us recall that we say that a hyperbolic curve X^{+} over a field k is *isotrivial* [cf. Definition 6.5] if, for an arbitrary separable closure $\overline{k}$ of k, there exist a hyperbolic curve X_0^{+} over the separable closure k_0 in $\overline{k}$ of the minimal subfield of k and an isomorphism $X^{+} \times_k \overline{k} \xrightarrow{\sim} X_0^{+} \times_{k_0} \overline{k}$ over $\overline{k}$. Then the classical anabelian results established by A. Tamagawa, S. Mochizuki, and J. Stix may be summarized as follows [cf. [30, Theorem 0.5], [18, Theorem 3.2], [28, Theorem 1], [29, Theorem 5.1.3]]:

Theorem A. *The following assertions hold:*

(i) *Suppose that both* $^{\dagger}k$ *and* $^{\ddagger}k$ *are finite. Write*

$$\mathrm{Isom}(^{\dagger}U, {}^{\ddagger}U)$$

for the set of isomorphisms $^{\dagger}U \xrightarrow{\sim} {}^{\ddagger}U$ *of schemes and*

$$\mathrm{OutIsom}(^{\dagger}\Pi, {}^{\ddagger}\Pi)$$

for the set of continuous outer isomorphisms $^{\dagger}\Pi \xrightarrow{\sim} {}^{\ddagger}\Pi$ *of topological groups. Then the natural map*

$$\mathrm{Isom}(^{\dagger}U, {}^{\ddagger}U) \xrightarrow{\sim} \mathrm{OutIsom}(^{\dagger}\Pi, {}^{\ddagger}\Pi)$$

is bijective.

(ii) *Suppose that the equality* $(^{\dagger}k, {}^{\dagger}\overline{k}) = (^{\ddagger}k, {}^{\ddagger}\overline{k})$ *holds, that* $^{\dagger}k$ *is finitely generated and transcendental over a finite field, and that the hyperbolic curve* $^{\dagger}X^{+}$ *is nonisotrivial. Write*

$$\mathrm{Isom}_{^{\dagger}k, F^{-1}_{^{\dagger}k}}(^{\dagger}U, {}^{\ddagger}U)$$

for the set of isomorphisms $^{\dagger}U \xrightarrow{\sim} {}^{\ddagger}U$ *in the category* $\mathsf{Var}_{{}^{\dagger}k, F_{{}^{\dagger}k}^{-1}}$ *defined in the discussion "Inverting Frobenius" following [29, Lemma 4.1.1] — i.e., obtained by forming the localization of the category of varieties over* $^{\dagger}k$ *and dominant morphisms over* $^{\dagger}k$ *with respect to universal homeomorphisms over* $^{\dagger}k$ *[which induce continuous isomorphisms between tame fundamental groups — cf., e.g., [34, Exposé IX, Théorème 4.10]] —*

$$\mathrm{Isom}_{G_{{}^{\dagger}k}}({}^{\dagger}\Pi, {}^{\ddagger}\Pi)$$

for the set of continuous isomorphisms $^{\dagger}\Pi \xrightarrow{\sim} {}^{\ddagger}\Pi$ *over* $G_{{}^{\dagger}k} = G_{{}^{\ddagger}k}$*, and*

$$_{\Delta}\backslash\mathrm{Isom}_{G_{{}^{\dagger}k}}({}^{\dagger}\Pi, {}^{\ddagger}\Pi)$$

for the quotient set of $\mathrm{Isom}_{G_{{}^{\dagger}k}}({}^{\dagger}\Pi, {}^{\ddagger}\Pi)$ *with respect to* $^{\ddagger}\Delta$*-conjugation. Then the natural map*

$$\mathrm{Isom}_{{}^{\dagger}k, F_{{}^{\dagger}k}^{-1}}({}^{\dagger}U, {}^{\ddagger}U) \xrightarrow{\sim} {}_{\Delta}\backslash\mathrm{Isom}_{G_{{}^{\dagger}k}}({}^{\dagger}\Pi, {}^{\ddagger}\Pi)$$

[cf. also [29, Corollary 4.2.5], the discussion following [29, Lemma 4.1.6]] is bijective.

Next, we introduce the notion of the *compactified configuration space* associated to a hyperbolic curve [cf. Definition 2.6]. Let n, g, r be nonnegative integers such that $2-2g-r<0$. Write $\overline{\mathscr{M}}_{g,n+r}$ for the moduli stack of $(n+r)$-pointed stable curves of genus g over $\mathbb{Z}$ [cf. [1, Proposition 5.1], [1, Theorem 5.2], [14, Theorem 2.7]] and $\mathscr{M}_{g,n+r} \subseteq \overline{\mathscr{M}}_{g,n+r}$ for the open substack of $\overline{\mathscr{M}}_{g,n+r}$ that parametrizes $(n+r)$-pointed stable curves of genus g whose underlying curves are smooth. Thus, we have a natural action of the symmetric group $\mathfrak{S}_{n+r}$ on $n+r$ letters on the algebraic stacks $\mathscr{M}_{g,n+r} \subseteq \overline{\mathscr{M}}_{g,n+r}$, i.e., that arises from the permutations of $n+r$ marked points. Write $\mathfrak{S}_{n+r,r} \subseteq \mathfrak{S}_{n+r}$ for the subgroup of $\mathfrak{S}_{n+r}$ of permutations of the last r letters, $\mathscr{M}_{g,n+[r]} \stackrel{\mathrm{def}}{=} [\mathscr{M}_{g,n+r}/\mathfrak{S}_{n+r,r}] \subseteq \overline{\mathscr{M}}_{g,n+[r]} \stackrel{\mathrm{def}}{=} [\overline{\mathscr{M}}_{g,n+r}/\mathfrak{S}_{n+r,r}]$ for the stack-theoretic quotients of the algebraic stacks $\mathscr{M}_{g,n+r} \subseteq \overline{\mathscr{M}}_{g,n+r}$ by the actions of the subgroup $\mathfrak{S}_{n+r,r} \subseteq \mathfrak{S}_{n+r}$ of $\mathfrak{S}_{n+r}$, respectively, and $\mathscr{D}_{g,n+[r]} \stackrel{\mathrm{def}}{=} (\overline{\mathscr{M}}_{g,n+[r]} \setminus \mathscr{M}_{g,n+[r]})_{\mathrm{red}} \subseteq \overline{\mathscr{M}}_{g,n+[r]}$ for the reduced closed substack of $\overline{\mathscr{M}}_{g,n+[r]}$ determined by the complement of $\mathscr{M}_{g,n+[r]}$ in $\overline{\mathscr{M}}_{g,n+[r]}$. Then one verifies immediately from the various definitions involved that if $n=0$, then the algebraic stack $\mathscr{M}_{g,n+[r]} = \mathscr{M}_{g,0+[r]}$ may be naturally identified with the moduli stack of hyperbolic curves of type (g,r) over $\mathbb{Z}$. Let S be a scheme, and let $X^{+} = (X, D)$ be a hyperbolic curve of type (g,r) over S. Then the n-th compactified configuration space of X^{+} is defined to be the pair

$$X^{+}_{(n)} \stackrel{\mathrm{def}}{=} (X_{(n)} \stackrel{\mathrm{def}}{=} \overline{\mathscr{M}}_{g,n+[r]} \times_{\overline{\mathscr{M}}_{g,0+[r]}} S,\ D^{X}_{(n)} \stackrel{\mathrm{def}}{=} \mathscr{D}_{g,n+[r]} \times_{\overline{\mathscr{M}}_{g,0+[r]}} S)$$

consisting of $X_{(n)}$, $D^{X}_{(n)}$ defined by the fiber products of the [representable — cf. [14, Corollary 2.6]] functors $\overline{\mathscr{M}}_{g,n+[r]} \to \overline{\mathscr{M}}_{g,0+[r]}$, $\mathscr{D}_{g,n+[r]} \to \overline{\mathscr{M}}_{g,0+[r]}$ obtained by forgetting the first n marked points and the classifying morphism $S \to \overline{\mathscr{M}}_{g,0+[r]}$ of the hyperbolic curve X^{+}, respectively. Observe that one verifies easily from the various definitions involved that the scheme over S

$$U^{X}_{(n)} \stackrel{\mathrm{def}}{=} X_{(n)} \setminus D^{X}_{(n)}$$

obtained by forming the complement of $D^{X}_{(n)}$ in $X_{(n)}$ may be naturally identified with the n-th configuration space of the curve $X \setminus D \subseteq X$ [cf., e.g., [21, Definition 2.1, (i)]]. Moreover, one also verifies easily from the various definitions involved that the scheme $X_{(n)}$ may be naturally identified with the underlying scheme of the n-th log configuration space of the curve $X \setminus D \subseteq X$ [cf., e.g., [4, Definition 1], [21, Definition 2.1, (i)]].

Now recall the notational conventions introduced in the discussion preceding Theorem A. For each $\square \in \{\dagger, \ddagger\}$, let ${}^{\square}n$ be a positive integer. Moreover, for each $\square \in \{\dagger, \ddagger\}$, write

– ${}^{\square}\Pi_{{}^{\square}n} \overset{\text{def}}{=} \pi_1^{\text{tame}}({}^{\square}X^{+}_{({}^{\square}n)})$ for the tame fundamental group of ${}^{\square}X^{+}_{({}^{\square}n)} \overset{\text{def}}{=} ({}^{\square}X_{({}^{\square}n)}, D^{{}^{\square}X}_{({}^{\square}n)})$, relative to a suitable choice of basepoint, and
– ${}^{\square}\Delta_{{}^{\square}n} \overset{\text{def}}{=} \pi_1^{\text{tame}}\big(({}^{\square}X^{+}_{{}^{\square}\overline{k}})_{({}^{\square}n)}\big)$ for the tame fundamental group of $({}^{\square}X^{+}_{{}^{\square}\overline{k}})_{({}^{\square}n)} \overset{\text{def}}{=} (({}^{\square}X_{{}^{\square}\overline{k}})_{({}^{\square}n)}, {}^{\square}D^{{}^{\square}X_{{}^{\square}\overline{k}}}_{({}^{\square}n)})$, relative to a suitable choice of basepoint.

Thus, for each $\square \in \{\dagger, \ddagger\}$, the natural morphisms $({}^{\square}X_{{}^{\square}\overline{k}})_{({}^{\square}n)} \to {}^{\square}X_{({}^{\square}n)} \to \operatorname{Spec}({}^{\square}k)$ determine an exact sequence of topological groups

$$1 \longrightarrow {}^{\square}\Delta_{{}^{\square}n} \longrightarrow {}^{\square}\Pi_{{}^{\square}n} \longrightarrow G_{{}^{\square}k} \longrightarrow 1.$$

Then our main result is as follows [cf. Theorem 6.7, Corollary 6.9]:

Theorem B. *The following assertions hold:*

(i) *Suppose that the following two conditions are satisfied:*

(i-1) *For each $\square \in \{\dagger, \ddagger\}$, the field ${}^{\square}k$ is the perfection of a field finitely generated over a finite field.*

(i-2) *If ${}^{\dagger}k$ is infinite, then the hyperbolic curve ${}^{\dagger}X^{+}$ is nonisotrivial.*

Write

$$\operatorname{Isom}(U^{{}^{\dagger}X}_{({}^{\dagger}n)}, U^{{}^{\ddagger}X}_{({}^{\ddagger}n)})$$

for the set of isomorphisms $U^{{}^{\dagger}X}_{({}^{\dagger}n)} \xrightarrow{\sim} U^{{}^{\ddagger}X}_{({}^{\ddagger}n)}$ of schemes and

$$\operatorname{OutIsom}({}^{\dagger}\Pi_{{}^{\dagger}n}, {}^{\ddagger}\Pi_{{}^{\ddagger}n})$$

for the set of continuous outer isomorphisms ${}^{\dagger}\Pi_{{}^{\dagger}n} \xrightarrow{\sim} {}^{\ddagger}\Pi_{{}^{\ddagger}n}$ of topological groups. Then the natural map

$$\operatorname{Isom}(U^{{}^{\dagger}X}_{({}^{\dagger}n)}, U^{{}^{\ddagger}X}_{({}^{\ddagger}n)}) \xrightarrow{\sim} \operatorname{OutIsom}({}^{\dagger}\Pi_{{}^{\dagger}n}, {}^{\ddagger}\Pi_{{}^{\ddagger}n})$$

[cf. also Proposition 6.2, (i)] is bijective.

(ii) *Suppose that the following three conditions are satisfied:*

(ii-1) *The equality $({}^{\dagger}k, {}^{\dagger}\overline{k}) = ({}^{\ddagger}k, {}^{\ddagger}\overline{k})$ holds.*

(ii-2) *The field ${}^{\dagger}k$ is finitely generated and transcendental over a finite field.*

(ii-3) *The hyperbolic curve ${}^{\dagger}X^{+}$ is nonisotrivial.*

Write

$$\operatorname{Isom}_{{}^{\dagger}k, F^{-1}_{{}^{\dagger}k}}(U^{{}^{\dagger}X}_{({}^{\dagger}n)}, U^{{}^{\ddagger}X}_{({}^{\ddagger}n)})$$

for the set of isomorphisms $U^{{}^{\dagger}X}_{({}^{\dagger}n)} \xrightarrow{\sim} U^{{}^{\ddagger}X}_{({}^{\ddagger}n)}$ in the category $\mathsf{Var}_{{}^{\dagger}k, F^{-1}_{{}^{\dagger}k}}$ defined in the discussion "Inverting Frobenius" following [29, Lemma 4.1.1],

$$\operatorname{Isom}_{G_{{}^{\dagger}k}}({}^{\dagger}\Pi_{{}^{\dagger}n}, {}^{\ddagger}\Pi_{{}^{\ddagger}n})$$

for the set of continuous isomorphisms ${}^{\dagger}\Pi_{{}^{\dagger}n} \overset{\sim}{\to} {}^{\ddagger}\Pi_{{}^{\ddagger}n}$ *over* $G_{{}^{\dagger}k} = G_{{}^{\ddagger}k}$, *and*

$$_{\Delta}\backslash\mathrm{Isom}_{G_{{}^{\dagger}k}}({}^{\dagger}\Pi_{{}^{\dagger}n}, {}^{\ddagger}\Pi_{{}^{\ddagger}n})$$

for the quotient set of $\mathrm{Isom}_{G_{{}^{\dagger}k}}({}^{\dagger}\Pi_{{}^{\dagger}n}, {}^{\ddagger}\Pi_{{}^{\ddagger}n})$ *with respect to* ${}^{\ddagger}\Delta_{{}^{\ddagger}n}$*-conjugation. Then the natural map*

$$\mathrm{Isom}_{{}^{\dagger}k, F^{-1}_{{}^{\dagger}k}}(U^{{}^{\dagger}X}_{({}^{\dagger}n)}, U^{{}^{\ddagger}X}_{({}^{\ddagger}n)}) \overset{\sim}{\longrightarrow} {}_{\Delta}\backslash\mathrm{Isom}_{G_{{}^{\dagger}k}}({}^{\dagger}\Pi_{{}^{\dagger}n}, {}^{\ddagger}\Pi_{{}^{\ddagger}n})$$

[*cf. also Theorem* 3.7, (ii); *Proposition* 6.2, (i); *the discussion following [29, Lemma 4.1.6]*] *is bijective.*

Theorem B may be regarded as a generalization of Theorem A. Moreover, Theorem B may also be regarded as the first anabelian Grothendieck conjecture-type result for algebraic varieties in positive characteristic of higher dimension [i.e., of dimension greater than one]. Here, we should emphasize that one may verify that a similar assertion to the assertion of Theorems A, B for the fiber products of finitely many hyperbolic curves over finite fields does not hold in general due to the existence of "incompatible Frobenius twists of the components of the fiber products under consideration" [cf. Remark 6.7.2, (i)]. This situation is totally different from the corresponding situation in characteristic zero [cf. Remark 6.7.2, (ii)]. In particular, it appears to the authors that this observation makes Theorem B interesting and difficult to find. Note that various anabelian Grothendieck conjecture-type results for the configuration spaces associated to hyperbolic curves over fields of characteristic zero have already been established [cf., e.g., [7, Theorem B], [7, Theorem 6.3]].

Next, observe that, in light of some results obtained in [32], one may easily derive Theorem B, (ii), from [the relative anabelian version — cf. Theorem 6.8 — of] Theorem B, (i) [cf. the proof of Corollary 6.9]. On the other hand, the key ingredients of the proof of Theorem B, (i), consist of the following results:

(a) Anabelian Grothendieck conjecture-type results for the geometrically "pro-prime-to-p" fundamental group of hyperbolic curves over [the perfections of] finitely generated fields of positive characteristic established by A. Tamagawa, M. Saïdi, and the third author of the present paper [cf. [24, Theorem 1], [25, Theorem D], [32, Theorem 2.9]].

(b) Certain exactness of homotopy sequences for the tame fundamental groups with respect to suitable morphisms between normal varieties [cf. §1, §2].

(c) Group-theoreticity of generalized fiber subgroups of the tame fundamental groups of the configuration spaces associated to hyperbolic curves [cf. §3, §4].

(d) Group-theoretic synchronization of cyclotomes that arise from the configuration spaces associated to hyperbolic curves [cf. §5].

Roughly speaking, (b) and (c) enable us to apply various anabelian Grothendieck conjecture-type results for hyperbolic curves [i.e., (a)] that arise from configuration spaces. Then (d) enables us to "control the Frobenius twists" of the hyperbolic curves that appear.

Our main theorem concerning (b) may be summarized as follows [cf. Lemma 1.8, (i); Theorem 1.13; Corollary 1.14]:

Theorem C. *Let*

– k *be a field,*
– $X^+ = (X, D_X)$ *and* $S^+ = (S, D_S)$ *good pairs over* k [*cf. Definition* 1.1],

– *$f\colon X \to S$ a morphism that is good with respect to (D_X, D_S) over k [cf. Definition 1.3], and*
– *$\overline{s} \to U_S$ a geometric point of $U_S \overset{\text{def}}{=} S \setminus D_S \subseteq S$.*

Write

– *$U_{X_{\overline{s}}}$ for the geometric fiber of $U_X \overset{\text{def}}{=} X \setminus D_X \subseteq X$ at $\overline{s} \to U_S$,*
– *$\underline{\pi_1^{\text{ét}}}(U_{X_{\overline{s}}})$ for the étale fundamental group $\pi_1^{\text{ét}}(U_{X_{\overline{s}}})$ of $U_{X_{\overline{s}}}$ (respectively, the maximal pro-prime-to-$\mathrm{char}(k)$ quotient of the étale fundamental group $\pi_1^{\text{ét}}(U_{X_{\overline{s}}})$ of $U_{X_{\overline{s}}}$), relative to a suitable choice of basepoint [cf. conditions (1), (3) of Definition 1.3], whenever the field k is of characteristic zero (respectively, of positive characteristic),*
– *$Z(\underline{\pi_1^{\text{ét}}}(U_{X_{\overline{s}}})) \subseteq \underline{\pi_1^{\text{ét}}}(U_{X_{\overline{s}}})$ for the center of $\underline{\pi_1^{\text{ét}}}(U_{X_{\overline{s}}})$,*
– *"$\pi_1^{\text{tame}}(-)$" for the tame fundamental group of the good pair "$(-)$", relative to a suitable choice of basepoint,*
– *$\Delta^{\text{tame}}_{X^+/S^+} \subseteq \pi_1^{\text{tame}}(X^+)$ for the kernel of the outer homomorphism $\pi_1^{\text{tame}}(X^+) \to \pi_1^{\text{tame}}(S^+)$ induced by f,*
– *$\underline{\Delta}^{\text{tame}}_{X^+/S^+} \overset{\text{def}}{=} \Delta^{\text{tame}}_{X^+/S^+}$ (respectively, $\underline{\Delta}^{\text{tame}}_{X^+/S^+}$ for the maximal pro-prime-to-$\mathrm{char}(k)$ quotient of $\Delta^{\text{tame}}_{X^+/S^+}$) whenever the field k is of characteristic zero (respectively, of positive characteristic), and*
– *$\underline{\Pi}^{\text{tame}}_{X^+/S^+}$ for the quotient of $\pi_1^{\text{tame}}(X^+)$ by $\mathrm{Ker}(\Delta^{\text{tame}}_{X^+/S^+} \twoheadrightarrow \underline{\Delta}^{\text{tame}}_{X^+/S^+})$. [Observe that since $\Delta^{\text{tame}}_{X^+/S^+}$ is normal in $\pi_1^{\text{tame}}(X^+)$, and $\mathrm{Ker}(\Delta^{\text{tame}}_{X^+/S^+} \twoheadrightarrow \underline{\Delta}^{\text{tame}}_{X^+/S^+})$ is characteristic in $\Delta^{\text{tame}}_{X^+/S^+}$, one verifies easily that $\mathrm{Ker}(\Delta^{\text{tame}}_{X^+/S^+} \twoheadrightarrow \underline{\Delta}^{\text{tame}}_{X^+/S^+})$ is normal in $\pi_1^{\text{tame}}(X^+)$.]*

Then the sequence of topological groups

$$\underline{\pi_1^{\text{ét}}}(U_{X_{\overline{s}}}) \longrightarrow \underline{\Pi}^{\text{tame}}_{X^+/S^+} \longrightarrow \pi_1^{\text{tame}}(S^+) \longrightarrow 1$$

induced by the natural morphisms $U_{X_{\overline{s}}} \to X \xrightarrow{f} S$ is exact. Moreover, the image of the kernel of the first arrow $\underline{\pi_1^{\text{ét}}}(U_{X_{\overline{s}}}) \to \underline{\Pi}^{\text{tame}}_{X^+/S^+}$ by the natural continuous surjective homomorphism $\underline{\pi_1^{\text{ét}}}(U_{X_{\overline{s}}}) \twoheadrightarrow \underline{\pi_1^{\text{ét}}}(U_{X_{\overline{s}}})/Z(\underline{\pi_1^{\text{ét}}}(U_{X_{\overline{s}}}))$ is contained in the center of the quotient $\underline{\pi_1^{\text{ét}}}(U_{X_{\overline{s}}})/Z(\underline{\pi_1^{\text{ét}}}(U_{X_{\overline{s}}}))$. In particular, if, moreover, the group $\underline{\pi_1^{\text{ét}}}(U_{X_{\overline{s}}})$ is center-free, then the exact sequence of topological groups

$$1 \longrightarrow \underline{\pi_1^{\text{ét}}}(U_{X_{\overline{s}}}) \longrightarrow \underline{\Pi}^{\text{tame}}_{X^+/S^+} \longrightarrow \pi_1^{\text{tame}}(S^+) \longrightarrow 1$$

induced by the natural morphisms $U_{X_{\overline{s}}} \to X \xrightarrow{f} S$ is exact.

By applying Theorem C to a projection morphism between configuration spaces [i.e., a morphism discussed in Definition 2.7, (ii), (iii)], one may conclude that a suitable quotient of the tame fundamental group of the configuration space admits a structure of an extension of the tame fundamental group of the configuration space of lower dimension associated to the given hyperbolic curve by the maximal "pro-prime-to-p" quotient of the étale fundamental group of the geometric fiber of the projection morphism [cf. Lemma 2.8, (iii)]. Now observe that, strictly speaking, in order to obtain the main theorem, we have to apply Theorem C to not only the configuration spaces but also various connected finite étale coverings of the configuration spaces [cf. the proof of Lemma 6.3].

Next, observe that, to conclude that such an extension structure of the tame fundamental group of the configuration space is group-theoretic, i.e., compatible with an arbitrary continuous isomorphisms, we have to prove that the subgroups of the tame fundamental groups of configuration spaces obtained by forming the kernels of the outer continuous homomorphisms induced by the projection morphisms, which we shall refer to as *generalized fiber subgroups* [cf. Definition 3.4], is group-theoretic. Our main theorem concerning (c) may be summarized as follows [cf. Corollary 4.9, (i), (ii)]:

Theorem D. *For each $\square \in \{\dagger, \ddagger\}$, let ${}^{\square}\Sigma$ be a nonempty set of prime numbers. Suppose, moreover, that, for each $\square \in \{\dagger, \ddagger\}$, the inequality $3 \leqslant \#{}^{\square}\Sigma$ holds whenever $\mathrm{char}({}^{\square}k) \in {}^{\square}\Sigma$. For each $\square \in \{\dagger, \ddagger\}$, write ${}^{\square}\Delta^{\square\Sigma}_{{}^{\square}n}$ for the maximal pro-${}^{\square}\Sigma$ quotient of the tame fundamental group ${}^{\square}\Delta_{{}^{\square}n}$. Let*

$$\alpha\colon {}^{\dagger}\Delta^{\dagger\Sigma}_{{}^{\dagger}n} \xrightarrow{\sim} {}^{\ddagger}\Delta^{\ddagger\Sigma}_{{}^{\ddagger}n}$$

be a continuous isomorphism. Then the equality ${}^{\dagger}n = {}^{\ddagger}n$ holds. Moreover, for an element i of $\{0, \ldots, {}^{\dagger}n = {}^{\ddagger}n\}$, the isomorphism α determines a bijective map between the set of generalized fiber subgroups of ${}^{\dagger}\Delta^{\dagger\Sigma}_{{}^{\dagger}n}$ of co-length i [cf. Definition 3.4] and the set of generalized fiber subgroups of ${}^{\ddagger}\Delta^{\ddagger\Sigma}_{{}^{\ddagger}n}$ of co-length i.

Note that a "group-theoretic reconstruction algorithm version" of Theorem D may be found in Theorem 4.8. Moreover, observe that Theorem D may be regarded as a generalization of [8, Theorem A], hence also of [21, Corollary 6.3]. More specifically, [8, Theorem A] is none other than Theorem D in the case where, for each $\square \in \{\dagger, \ddagger\}$, the set ${}^{\square}\Sigma$ consists either of all prime numbers or of a single prime number invertible in ${}^{\square}k$.

By applying suitable anabelian Grothendieck conjecture-type results for hyperbolic curves [i.e., (a)] to suitable generalized fiber subgroups [equipped with suitable outer Galois actions], one obtains isomorphisms of the hyperbolic curves [i.e., obtained by forming the geometric fiber of the projection morphism] "up to Frobenius twists". In particular, to obtain an isomorphism of the configuration spaces of the desired type, it suffices to "control the Frobenius twists". This control/compatibility of Frobenius twists may be interpreted as the phenomenon of group-theoretic synchronization of cyclotomes associated to the hyperbolic curves that appear, which is consistent with the viewpoint of [32]. This step is established in detail in §5, which is the content of (d) [cf. Lemma 5.5, (vi)]. Finally, in §6, by combining the above results, we complete the proof of Theorem B, (i).

In Appendix A, following a request from the referee, we give a quick review of the purely "group-theoretic algorithm" that constructs the generalized fiber subgroups of configuration space groups established in [8] for convenience of readers.

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1. Homotopy sequences for good pairs

In the present §1, we study the homotopy sequences for the tame fundamental groups of suitable morphisms between normal varieties [cf. Corollary 1.14 below].

Definition 1.1. Let k be a field.

(i) We shall define a *normal variety* over k to be a scheme that is separated, of finite type, geometrically connected, and geometrically normal over k.

(ii) We shall define a *good pair* over k to be a pair consisting of a normal [cf. Remark 1.1.1 below] variety X over k and a reduced closed subscheme $D \subseteq X$ of X of pure codimension one such that the [necessarily fs] log scheme obtained by equipping X with the log structure determined by D is log smooth over k.

(iii) Let $X^+ = (X, D)$ be a good pair over k. Thus, it is well-known [cf., e.g., [5, Proposition B.7], [5, Theorem B.1]] that the category of finite étale coverings of $X \setminus D$ tamely ramified along D forms a Galois category. We shall write

$$\pi_1^{\mathrm{tame}}(X^+) = \pi_1^{\mathrm{tame}}(X, D)$$

for the Galois group of this Galois category, relative to a suitable choice of basepoint [i.e., of fiber functor], and refer to $\pi_1^{\mathrm{tame}}(X^+)$ as the *tame fundamental group* of the good pair (X, D). Moreover, we shall write

$$\pi_1^{\text{ét}}(X)$$

for the étale fundamental group of the normal variety X, relative to a suitable choice of basepoint.

Remark 1.1.1. Let us recall [cf., e.g., [5, Proposition A.3], [5, Proposition A.5]] that the underlying scheme of an fs log scheme that is log smooth over a field k is log regular and geometrically normal over the field k.

Lemma 1.2. *Let k be a field, (X, D) a good pair over k, and $V \to X \setminus D$ a connected finite étale covering tamely ramified along D. Write K for the algebraic closure of k in the function field of V, $Y \to X$ for the normalization of X in V, and $E \overset{\mathrm{def}}{=} (Y \setminus V)_{\mathrm{red}} \subseteq Y$ for the reduced closed subscheme of Y whose underlying closed subset is given by $Y \setminus V$. Write $X^{\log}$, $Y^{\log}$ for the [necessarily fs] log schemes obtained by equipping X, Y with the log structures determined by D, E, respectively. Then the following assertions hold:*

(i) *The natural morphism $Y^{\log} \to X^{\log}$ is a connected Kummer finite log étale covering.*

(ii) *The pair (Y, E) is a good pair over K.*

Proof. Assertion (i) follows from [5, Proposition B.7] [cf. also [5, Remark B.2]]. Assertion (ii) follows from assertion (i). □

Definition 1.3. Let k be a field, and let (X, D_X), (S, D_S) be good pairs over k. Then we shall say that a morphism $f \colon X \to S$ over k is *good with respect to* (D_X, D_S) if the following five conditions are satisfied:

(1) The morphism f is proper and geometrically connected.

(2) The inclusion $f^{-1}D_S \subseteq D_X$ holds.

(3) The morphism $X \setminus f^{-1}D_S \to S \setminus D_S$ determined by f [cf. (2)] is smooth.

(4) The composite $D_X \cap (X \setminus f^{-1}D_S) \to S \setminus D_S$ of the natural closed immersion $D_X \cap (X \setminus f^{-1}D_S) \hookrightarrow X \setminus f^{-1}D_S$ with the morphism $X \setminus f^{-1}D_S \to S \setminus D_S$ determined by f [cf. (2)] is smooth and of pure relative codimension one.

(5) If one writes $X^{\log}$, $S^{\log}$ for the [necessarily fs and log regular] log schemes obtained by equipping X, S with the log structures determined by D_X, D_S, respectively, then the morphism $X^{\log} \to S^{\log}$ determined by f [cf. (2)] is log smooth.

Lemma 1.4. *Let k be a field, (X, D_X) and (S, D_S) good pairs over k, $f\colon X \to S$ a morphism over k that is good with respect to (D_X, D_S), and $\overline{s} \to S \setminus D_S$ a geometric point of $S \setminus D_S$. Write $X_{\overline{s}}$, $D_{X_{\overline{s}}}$ for the geometric fibers of X, D_X at $\overline{s} \to S \setminus D_S$, respectively. Then the pair $(X_{\overline{s}}, D_{X_{\overline{s}}})$ is a good pair over $\overline{s}$.*

Proof. This assertion follows from conditions (1), (5) of Definition 1.3 [cf. also Remark 1.1.1]. □

Here, let us recall the notion of log Stein factorization [cf. [5, Definition 3]]. Let $X^{\log}$ and $Y^{\log}$ be log regular log schemes. Then every proper log smooth morphism $Y^{\log} \to X^{\log}$ admits the log Stein factorization [cf. [5, Theorem 1]], which is a factorization $Y^{\log} \to Z^{\log} \to X^{\log}$ of the given morphism $Y^{\log} \to X^{\log}$ that satisfies the following two conditions:

- The morphism $Z^{\log} \to X^{\log}$ is a Kummer finite log étale covering.
- The morphism $Y^{\log} \to Z^{\log}$ is log geometrically connected [cf. [5, Definition 2]].

The next lemma discusses the notion of log Stein factorization. Moreover, we will apply the next lemma to prove the exactness of the homotopy sequence associated to a good morphism between good pairs.

Lemma 1.5. *Let k be a field, (X, D_X) and (S, D_S) good pairs over k, $f\colon X \to S$ a morphism over k that is good with respect to (D_X, D_S), and $V \to X \setminus D_X$ a connected finite étale covering tamely ramified along D_X. Write K for the algebraic closure of k in the function field of V, $Y \to X$ for the normalization of X in V, and $D_Y \stackrel{\mathrm{def}}{=} (Y \setminus V)_{\mathrm{red}} \subseteq Y$ for the reduced closed subscheme of Y whose underlying closed subset is given by $Y \setminus V$. Thus, it follows from Lemma 1.2,* (ii), *that the pair (Y, D_Y) is a good pair over K. Write $X^{\log}$, $S^{\log}$, $Y^{\log}$ for the [necessarily fs and log regular] log schemes obtained by equipping X, S, Y with the log structures determined by D_X, D_S, D_Y, respectively. Then the following assertions hold:*

(i) *The composite $Y^{\log} \to X^{\log} \to S^{\log}$ is a log smooth morphism whose underlying morphism of schemes is proper.*

(ii) *Write $Y^{\log} \to T^{\log} \to S^{\log}$ for the log Stein factorization [cf. [5, Definition 3]] of the composite $Y^{\log} \to X^{\log} \to S^{\log}$ [cf.* (i)*]. Then the natural morphism $T^{\log} \to S^{\log}$ is a connected Kummer finite log étale covering.*

(iii) *Write T for the underlying scheme of $T^{\log}$, $U_T \stackrel{\mathrm{def}}{=} T \times_S (S \setminus D_S) \subseteq T$, and $D_T \stackrel{\mathrm{def}}{=} (T \setminus U_T)_{\mathrm{red}} \subseteq T$ for the reduced closed subscheme whose underlying closed subset is given by $T \setminus U_T$. Then the pair (T, D_T) is a good pair over K.*

(iv) *The scheme T is isomorphic, over S, to the normalization of S in Y.*

(v) *The natural morphism $Y \to T$ over K is good with respect to (D_Y, D_T) [cf.* (iii)*].*

Proof. Assertion (i) follows from Lemma 1.2, (i), together with conditions (1), (5) of Definition 1.3. Assertion (ii) is a formal consequence of [5, Theorem 1, (i)]. Assertion (iii) follows immediately from assertion (ii). Assertion (iv) follows immediately from the definition of the notion of the log Stein factorization.

Finally, we verify assertion (v). It follows from assertion (i) that the morphism $Y \to T$ is proper. Moreover, it follows from [5, Theorem 1, (ii)] that the morphism $Y \to T$ is geometrically connected. This completes the proof of the assertion that the morphism $Y \to T$ satisfies condition (1) of Definition 1.3. Next, observe that it is immediate that the morphism $Y \to T$ satisfies condition (2) of Definition 1.3. Next, observe that one verifies immediately, by considering the morphisms $Y^{\log} \to T^{\log} \to S^{\log}$, from [13, Proposition 3.12], together with assertions (i), (ii), that the morphism $Y \to T$ satisfies condition (5) of Definition 1.3.

Next, we verify the assertion that the morphism $Y \to T$ satisfies condition (4) of Definition 1.3. First, observe that since the morphism $T^{\log} \to S^{\log}$ is log étale [cf. assertion (ii)], it follows from [13, Proposition 3.8] that the induced morphism $U_T \to S \setminus D_S$ is étale. Thus, since [we have assumed that] the morphism $X \to S$ satisfies condition (4) of Definition 1.3, one verifies immediately from Abhyankar's lemma [cf. [34, Exposé XIII, Proposition 5.5]], together with Lemma 1.2, (i), that the morphism $Y \to T$ satisfies condition (4) of Definition 1.3. This completes the proof of the assertion that the morphism $Y \to T$ satisfies condition (4) of Definition 1.3.

Next, we verify the assertion that the morphism $Y \to T$ satisfies condition (3) of Definition 1.3. Let us recall that the morphism $Y^{\log} \to T^{\log}$ is log smooth [cf. condition (5) of Definition 1.3]. Thus, since the morphism $Y \to T$ satisfies condition (4) of Definition 1.3, it follows immediately from [13, Theorem 3.5] that the morphism $Y \to T$ satisfies condition (3) of Definition 1.3. This completes the proof of the assertion that the morphism $Y \to T$ satisfies condition (3) of Definition 1.3, hence also of assertion (v). □

In the remainder of the present §1, let

- k be a field,
- $X^+ = (X, D_X)$ and $S^+ = (S, D_S)$ good pairs over k, and
- $f\colon X \to S$ a morphism which is good with respect to (D_X, D_S) over k.

Write

- $U_X \overset{\text{def}}{=} X \setminus D_X \subseteq X$ and
- $U_S \overset{\text{def}}{=} S \setminus D_S \subseteq S$.

Definition 1.6.

(i) We shall write

$$\pi_1^{\text{ét}}(f)\colon \pi_1^{\text{ét}}(U_X) \longrightarrow \pi_1^{\text{ét}}(U_S)$$

for the continuous outer homomorphism induced by the morphism $f\colon X \to S$ [cf. condition (2) of Definition 1.3],

$$\Delta_{U_X/U_S} \overset{\text{def}}{=} \operatorname{Ker}\bigl(\pi_1^{\text{ét}}(f)\colon \pi_1^{\text{ét}}(U_X) \to \pi_1^{\text{ét}}(U_S)\bigr) \subseteq \pi_1^{\text{ét}}(U_X)$$

for the kernel of the outer homomorphism $\pi_1^{\text{ét}}(f)\colon \pi_1^{\text{ét}}(U_X) \to \pi_1^{\text{ét}}(U_S)$,

$$\underline{\Delta}_{U_X/U_S} \quad (\twoheadleftarrow \Delta_{U_X/U_S})$$

for the identity quotient, i.e., $\underline{\Delta}_{U_X/U_S} \stackrel{\text{def}}{=} \Delta_{U_X/U_S}$ (respectively, for the maximal pro-prime-to-char(k) quotient of Δ_{U_X/U_S}), whenever the field k is of characteristic zero (respectively, of positive characteristic), and

$$\underline{\Pi}_{U_X/U_S} \stackrel{\text{def}}{=} \pi_1^{\text{ét}}(U_X)/\text{Ker}(\Delta_{U_X/U_S} \twoheadrightarrow \underline{\Delta}_{U_X/U_S}).$$

[Observe that since Δ_{U_X/U_S} is normal in $\pi_1^{\text{ét}}(U_X)$, and $\text{Ker}(\Delta_{U_X/U_S} \twoheadrightarrow \underline{\Delta}_{U_X/U_S})$ is characteristic in Δ_{U_X/U_S}, one verifies easily that $\text{Ker}(\Delta_{U_X/U_S} \twoheadrightarrow \underline{\Delta}_{U_X/U_S})$ is normal in $\pi_1^{\text{ét}}(U_X)$.] By a slight abuse of notation, we shall write

$$\pi_1^{\text{ét}}(f)\colon \underline{\Pi}_{U_X/U_S} \longrightarrow \pi_1^{\text{ét}}(U_S)$$

for the continuous outer homomorphism determined by $\pi_1^{\text{ét}}(f)\colon \pi_1^{\text{ét}}(U_X) \to \pi_1^{\text{ét}}(U_S)$. Thus, we have an exact sequence of topological groups

$$1 \longrightarrow \underline{\Delta}_{U_X/U_S} \longrightarrow \underline{\Pi}_{U_X/U_S} \xrightarrow{\pi_1^{\text{ét}}(f)} \pi_1^{\text{ét}}(U_S).$$

(ii) We shall write

$$\pi_1^{\text{tame}}(f)\colon \pi_1^{\text{tame}}(X^+) \longrightarrow \pi_1^{\text{tame}}(S^+)$$

for the continuous outer homomorphism induced by the morphism $f\colon X \to S$ [cf. condition (2) of Definition 1.3],

$$\Delta_{X^+/S^+}^{\text{tame}} \stackrel{\text{def}}{=} \text{Ker}\big(\pi_1^{\text{tame}}(f)\colon \pi_1^{\text{tame}}(X^+) \to \pi_1^{\text{tame}}(S^+)\big) \subseteq \pi_1^{\text{tame}}(X^+)$$

for the kernel of the outer homomorphism $\pi_1^{\text{tame}}(f)\colon \pi_1^{\text{tame}}(X^+) \to \pi_1^{\text{tame}}(S^+)$,

$$\underline{\Delta}_{X^+/S^+}^{\text{tame}} \quad (\twoheadleftarrow \Delta_{X^+/S^+}^{\text{tame}})$$

for the identity quotient, i.e., $\underline{\Delta}_{X^+/S^+}^{\text{tame}} \stackrel{\text{def}}{=} \Delta_{X^+/S^+}^{\text{tame}}$ (respectively, for the maximal pro-prime-to-char(k) quotient of $\Delta_{X^+/S^+}^{\text{tame}}$), whenever the field k is of characteristic zero (respectively, of positive characteristic), and

$$\underline{\Pi}_{X^+/S^+}^{\text{tame}} \stackrel{\text{def}}{=} \pi_1^{\text{tame}}(X^+)/\text{Ker}(\Delta_{X^+/S^+}^{\text{tame}} \twoheadrightarrow \underline{\Delta}_{X^+/S^+}^{\text{tame}}).$$

[Observe that since $\Delta_{X^+/S^+}^{\text{tame}}$ is normal in $\pi_1^{\text{tame}}(X^+)$, and $\text{Ker}(\Delta_{X^+/S^+}^{\text{tame}} \twoheadrightarrow \underline{\Delta}_{X^+/S^+}^{\text{tame}})$ is characteristic in $\Delta_{X^+/S^+}^{\text{tame}}$, one verifies easily that $\text{Ker}(\Delta_{X^+/S^+}^{\text{tame}} \twoheadrightarrow \underline{\Delta}_{X^+/S^+}^{\text{tame}})$ is normal in $\pi_1^{\text{tame}}(X^+)$.] By a slight abuse of notation, we shall write

$$\pi_1^{\text{tame}}(f)\colon \underline{\Pi}_{X^+/S^+}^{\text{tame}} \longrightarrow \pi_1^{\text{tame}}(S^+)$$

for the continuous outer homomorphism determined by $\pi_1^{\text{tame}}(f)\colon \pi_1^{\text{tame}}(X^+) \to \pi_1^{\text{tame}}(S^+)$. Thus, we have an exact sequence of topological groups

$$1 \longrightarrow \underline{\Delta}_{X^+/S^+}^{\text{tame}} \longrightarrow \underline{\Pi}_{X^+/S^+}^{\text{tame}} \xrightarrow{\pi_1^{\text{tame}}(f)} \pi_1^{\text{tame}}(S^+).$$

In the remainder of the present §1, let $\overline{s} \to U_S$ be a geometric point of U_S. Write

– $X_{\overline{s}}$, $D_{X_{\overline{s}}}$, $U_{X_{\overline{s}}}$ for the geometric fibers of X, D_X, U_X at $\overline{s} \to U_S$, respectively, and

– $\underline{\pi_1^{\text{ét}}}(U_{X_{\overline{s}}}) \overset{\text{def}}{=} \pi_1^{\text{ét}}(U_{X_{\overline{s}}})$ (respectively, $\underline{\pi_1^{\text{ét}}}(U_{X_{\overline{s}}})$ for the maximal pro-prime-to-char(k) quotient of $\pi_1^{\text{ét}}(U_{X_{\overline{s}}})$) [cf. conditions (1), (3) of Definition 1.3] whenever the field k is of characteristic zero (respectively, of positive characteristic).

Lemma 1.7. *The $\pi_1^{\text{tame}}(X^+)$-conjugacy class of continuous homomorphisms*

$$\pi_1^{\text{tame}}(X^+_{\overline{s}}) \longrightarrow \Delta^{\text{tame}}_{X^+/S^+}$$

[cf. Lemma 1.4] induced by the natural morphism $X_{\overline{s}} \to X$ is surjective.

Proof. This assertion follows from [5, Theorem 2] and [5, Proposition B.7] [cf. conditions (1), (5) of Definition 1.3]. □

Lemma 1.8. *Consider the commutative diagram of topological groups*

$$\begin{array}{ccccccc}
\underline{\pi_1^{\text{ét}}}(U_{X_{\overline{s}}}) & \longrightarrow & \underline{\Pi}_{U_X/U_S} & \xrightarrow{\pi_1^{\text{ét}}(f)} & \pi_1^{\text{ét}}(U_S) & \longrightarrow & 1 \\
\| & & \downarrow & & \downarrow & & \\
\underline{\pi_1^{\text{ét}}}(U_{X_{\overline{s}}}) & \longrightarrow & \underline{\Pi}^{\text{tame}}_{X^+/S^+} & \xrightarrow[\pi_1^{\text{tame}}(f)]{} & \pi_1^{\text{tame}}(S^+) & \longrightarrow & 1
\end{array}$$

— where the left-hand horizontal arrows are the continuous outer homomorphisms induced by the natural morphism $U_{X_{\overline{s}}} \to U_X$, and the vertical arrows are the natural continuous surjective homomorphisms. Then the following assertions hold:

(i) *The two horizontal sequences of the diagram under consideration are exact.*

(ii) *The kernel of the left-hand upper horizontal arrow $\underline{\pi_1^{\text{ét}}}(U_{X_{\overline{s}}}) \to \underline{\Pi}_{U_X/U_S}$ of the diagram under consideration is contained in the center of $\underline{\pi_1^{\text{ét}}}(U_{X_{\overline{s}}})$.*

Proof. The exactness of the upper horizontal sequence of the diagram under consideration follows from [2, Proposition 1.3] [cf. conditions (1), (3), (4) of Definition 1.3]. The exactness of the lower horizontal sequence of the diagram under consideration follows from [5, Theorem 2] and [5, Proposition B.7] [cf. conditions (1), (5) of Definition 1.3]. Assertion (ii) follows from [2, Proposition 1.4] [cf. conditions (1), (3), (4) of Definition 1.3]. □

The remainder of the present §1 is devoted to studying the kernel of the left-hand lower horizontal arrow $\underline{\pi_1^{\text{ét}}}(U_{X_{\overline{s}}}) \to \underline{\Pi}^{\text{tame}}_{X^+/S^+}$ of the diagram of Lemma 1.8 and establishing some conditions under which the homomorphism $\underline{\pi_1^{\text{ét}}}(U_{X_{\overline{s}}}) \to \underline{\Pi}^{\text{tame}}_{X^+/S^+}$ under consideration is injective.

Definition 1.9. Observe that it follows from Lemma 1.8, (i), that we have an exact sequence of topological groups

$$1 \longrightarrow \underline{\Delta}_{U_X/U_S} \longrightarrow \underline{\Pi}_{U_X/U_S} \xrightarrow{\pi_1^{\text{ét}}(f)} \pi_1^{\text{ét}}(U_S) \longrightarrow 1.$$

We shall write

$$\rho_{\overline{s}} \colon \pi_1^{\text{ét}}(U_S) \longrightarrow \operatorname{Out}(\underline{\Delta}_{U_X/U_S})$$

for the continuous outer action determined by this exact sequence of topological groups.

Lemma 1.10. *The following assertions hold:*

(i) *The image of a wild inertia subgroup of $\pi_1^{\text{ét}}(U_S)$ associated to an irreducible component of the closed subscheme D_S by the continuous outer action $\rho_{\overline{s}} \colon \pi_1^{\text{ét}}(U_S) \to \operatorname{Out}(\underline{\Delta}_{U_X/U_S})$ is trivial.*

(ii) *The continuous outer action* $\rho_{\overline{s}}\colon \pi_1^{\text{ét}}(U_S) \to \mathrm{Out}(\underline{\Delta}_{U_X/U_S})$ *factors through the natural continuous surjective homomorphism* $\pi_1^{\text{ét}}(U_S) \twoheadrightarrow \pi_1^{\text{tame}}(S^+)$.

Proof. This assertion follows immediately — in light of conditions (1), (5) of Definition 1.3 — from [29, Corollary 3.4.7] [i.e., [33, Chapter I, Proposition 3.2]]. □

Definition 1.11. We shall write

$$\rho_{\overline{s}}^{\text{tame}}\colon \pi_1^{\text{tame}}(S^+) \longrightarrow \mathrm{Out}(\underline{\Delta}_{U_X/U_S})$$

for the continuous outer action determined by the continuous outer action $\rho_{\overline{s}}\colon \pi_1^{\text{ét}}(U_S) \to \mathrm{Out}(\underline{\Delta}_{U_X/U_S})$ [cf. Lemma 1.10, (ii)] and

$$\underline{E}(U_{X_{\overline{s}}}) \overset{\text{def}}{=} \mathrm{Aut}(\underline{\Delta}_{U_X/U_S}) \times_{\mathrm{Out}(\underline{\Delta}_{U_X/U_S})} \pi_1^{\text{tame}}(S^+)$$

for the fiber product of the natural surjective homomorphism $\mathrm{Aut}(\underline{\Delta}_{U_X/U_S}) \twoheadrightarrow \mathrm{Out}(\underline{\Delta}_{U_X/U_S})$ and the outer action $\rho_{\overline{s}}^{\text{tame}}\colon \pi_1^{\text{tame}}(S^+) \to \mathrm{Out}(\underline{\Delta}_{U_X/U_S})$. Thus, the natural exact sequence of groups

$$\underline{\Delta}_{U_X/U_S} \longrightarrow \mathrm{Aut}(\underline{\Delta}_{U_X/U_S}) \longrightarrow \mathrm{Out}(\underline{\Delta}_{U_X/U_S}) \longrightarrow 1$$

— where the first arrow is a continuous action by conjugation — determines an exact sequence of groups

$$1 \longrightarrow \underline{\Delta}_{U_X/U_S}/Z(\underline{\Delta}_{U_X/U_S}) \longrightarrow \underline{E}(U_{X_{\overline{s}}}) \longrightarrow \pi_1^{\text{tame}}(S^+) \longrightarrow 1.$$

Lemma 1.12. *The following assertions hold:*

(i) *The continuous action* $\underline{\Pi}_{U_X/U_S} \to \mathrm{Aut}(\underline{\Delta}_{U_X/U_S})$ *by conjugation* [*cf. the displayed exact sequence of Definition* 1.9] *and the natural continuous surjective homomorphisms* $\underline{\Pi}_{U_X/U_S} \twoheadrightarrow \pi_1^{\text{ét}}(U_S) \twoheadrightarrow \pi_1^{\text{tame}}(S^+)$ *determine a commutative diagram of groups*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \underline{\Delta}_{U_X/U_S} & \longrightarrow & \underline{\Pi}_{U_X/U_S} & \xrightarrow{\pi_1^{\text{ét}}(f)} & \pi_1^{\text{ét}}(U_S) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \underline{\Delta}_{U_X/U_S}/Z(\underline{\Delta}_{U_X/U_S}) & \longrightarrow & \underline{E}(U_{X_{\overline{s}}}) & \longrightarrow & \pi_1^{\text{tame}}(S^+) & \longrightarrow & 1 \end{array}$$

— *where the upper horizontal sequence is the displayed exact sequence of Definition* 1.9, *the lower horizontal sequence is the exact sequence of the final display of Definition* 1.11, *and the left-hand and right-hand vertical arrows are the natural continuous surjective homomorphisms* [*which thus implies that the middle vertical arrow is surjective*].

(ii) *The middle vertical arrow* $\underline{\Pi}_{U_X/U_S} \twoheadrightarrow \underline{E}(U_{X_{\overline{s}}})$ *of the diagram of* (i) *factors through the natural continuous surjective homomorphism* $\underline{\Pi}_{U_X/U_S} \twoheadrightarrow \underline{\Pi}_{X^+/S^+}^{\text{tame}}$.

Proof. Assertion (i) follows immediately from the various definitions involved. Next, we verify assertion (ii). Let us first observe that it is immediate that, to verify assertion (ii), it suffices to verify that the image of every wild inertia subgroup of $\pi_1^{\text{ét}}(U_X)$ associated to an irreducible component of the closed subscheme D_X by the composite $\pi_1^{\text{ét}}(U_X) \twoheadrightarrow \underline{\Pi}_{U_X/U_S} \twoheadrightarrow \underline{E}(U_{X_{\overline{s}}})$ is trivial. Let $P \subseteq \pi_1^{\text{ét}}(U_X)$ be a wild inertia subgroup of $\pi_1^{\text{ét}}(U_X)$ associated to an irreducible component of the closed subscheme D_X. Then it follows immediately from the various definitions involved that the image of $P \subseteq \pi_1^{\text{ét}}(U_X)$ in $\pi_1^{\text{tame}}(S^+)$ is trivial. In particular, it follows from assertion (i) that the image of $P \subseteq \pi_1^{\text{ét}}(U_X)$ by the composite $\pi_1^{\text{ét}}(U_X) \twoheadrightarrow \underline{\Pi}_{U_X/U_S} \twoheadrightarrow \underline{E}(U_{X_{\overline{s}}})$ is contained in the closed subgroup $\underline{\Delta}_{U_X/U_S}/Z(\underline{\Delta}_{U_X/U_S}) \subseteq \underline{E}(U_{X_{\overline{s}}})$ of $\underline{E}(U_{X_{\overline{s}}})$. Thus, since $\underline{\Delta}_{U_X/U_S}/Z(\underline{\Delta}_{U_X/U_S})$ is pro-prime-to-char(k) whenever the field k is of positive characteristic, one concludes that the image of $P \subseteq \pi_1^{\text{ét}}(U_X)$ by the composite $\pi_1^{\text{ét}}(U_X) \twoheadrightarrow \underline{\Pi}_{U_X/U_S} \twoheadrightarrow \underline{E}(U_{X_{\overline{s}}})$ is trivial, as desired. This completes the proof of assertion (ii), hence also of Lemma 1.12. □

THEOREM 1.13. *The image of the kernel of the left-hand lower horizontal arrow* $\underline{\pi_1^{\text{ét}}}(U_{X_{\overline{s}}}) \to \underline{\Pi}^{\text{tame}}_{X^+/S^+}$ *of the diagram of the statement of Lemma* 1.8 *by the natural continuous surjective homomorphism* $\underline{\pi_1^{\text{ét}}}(U_{X_{\overline{s}}}) \twoheadrightarrow \underline{\pi_1^{\text{ét}}}(U_{X_{\overline{s}}})/Z(\underline{\pi_1^{\text{ét}}}(U_{X_{\overline{s}}}))$ *is contained in the center of the quotient* $\underline{\pi_1^{\text{ét}}}(U_{X_{\overline{s}}})/Z(\underline{\pi_1^{\text{ét}}}(U_{X_{\overline{s}}}))$.

Proof. Let us first observe that it follows immediately from the various definitions involved that the left-hand lower horizontal arrow $\underline{\pi_1^{\text{ét}}}(U_{X_{\overline{s}}}) \to \underline{\Pi}^{\text{tame}}_{X^+/S^+}$ of the diagram of the statement of Lemma 1.8 factors as the composite

$$\underline{\pi_1^{\text{ét}}}(U_{X_{\overline{s}}}) \twoheadrightarrow \underline{\Delta}_{U_X/U_S} \longrightarrow \underline{\Pi}^{\text{tame}}_{X^+/S^+}.$$

Next, let us recall from Lemma 1.8, (ii), that the kernel of the first arrow $\underline{\pi_1^{\text{ét}}}(U_{X_{\overline{s}}}) \twoheadrightarrow \underline{\Delta}_{U_X/U_S}$ is contained in the center of $\underline{\pi_1^{\text{ét}}}(U_{X_{\overline{s}}})$. Moreover, let us recall from Lemma 1.12, (i), (ii), that the kernel of the second arrow $\underline{\Delta}_{U_X/U_S} \to \underline{\Pi}^{\text{tame}}_{X^+/S^+}$ is contained in the center of $\underline{\Delta}_{U_X/U_S}$. Thus, the desired assertion follows formally. This completes the proof of Theorem 1.13. □

COROLLARY 1.14. *Suppose that the group* $\underline{\pi_1^{\text{ét}}}(U_{X_{\overline{s}}})$ *is center-free. Then the natural morphism* $X_{\overline{s}} \to X$ *and the morphism* $f\colon X \to S$ *determine an exact sequence of topological groups*

$$1 \longrightarrow \underline{\pi_1^{\text{ét}}}(U_{X_{\overline{s}}}) \longrightarrow \underline{\Pi}^{\text{tame}}_{X^+/S^+} \xrightarrow{\pi_1^{\text{tame}}(f)} \pi_1^{\text{tame}}(S^+) \longrightarrow 1.$$

Put another way, the natural morphism $X_{\overline{s}} \to X$ *determines a* $\underline{\Pi}^{\text{tame}}_{X^+/S^+}$*-conjugacy class of continuous isomorphisms*

$$\underline{\pi_1^{\text{ét}}}(U_{X_{\overline{s}}}) \xrightarrow{\sim} \underline{\Delta}^{\text{tame}}_{X^+/S^+}.$$

Proof. This assertion is a formal consequence of Lemma 1.8 and Theorem 1.13. □

2. Compactified configuration spaces of hyperbolic curves

In the present §2, we introduce the notion of the compactified configuration space of a hyperbolic curve [cf. Definition 2.6 below]. Moreover, we study the homotopy sequences for the tame fundamental groups of connected tamely ramified finite coverings of compactified configuration spaces [cf. Lemma 2.8, (iii), below].

In the present §2, let g, r be nonnegative integers such that $2 - 2g - r < 0$.

DEFINITION 2.1. Let S be a scheme. Then we shall define a *hyperbolic curve of type* (g, r) over S to be a pair (X, D) consisting of a scheme X over S and a closed subscheme $D \subseteq X$ of X such that

- the scheme X is proper, geometrically connected, smooth, and of relative dimension one over S,
- every geometric fiber of X over S is [a necessarily smooth and proper curve] of genus g, and, moreover,
- the composite $D \hookrightarrow X \to S$ is étale and of degree r.

We shall define a *hyperbolic curve* over S to be a hyperbolic curve of type (g', r') some nonnegative integers g', r' such that $2 - 2g' - r' < 0$.

Definition 2.2.

(i) We shall write

$$\mathfrak{S}_r$$

for the *symmetric group on* r *letters.*

(ii) We shall write

$$\overline{\mathscr{M}}_{g,r}$$

for the *moduli stack of* r*-pointed stable curves of genus* g over $\mathbb{Z}$ [cf. [1, Proposition 5.1], [1, Theorem 5.2], [14, Theorem 2.7]] and

$$\mathscr{M}_{g,r} \subseteq \overline{\mathscr{M}}_{g,r}$$

for the open substack of $\overline{\mathscr{M}}_{g,r}$ that parametrizes r-pointed stable curves of genus g whose underlying curves are smooth.

Definition 2.3. Let n be a nonnegative integer.

(i) We shall write

$$\mathfrak{S}_{n+r,r} \subseteq \mathfrak{S}_{n+r}$$

for the subgroup of $\mathfrak{S}_{n+r}$ [necessarily isomorphic to $\mathfrak{S}_r$] of permutations of the last r letters.

(ii) Observe that one verifies easily that we have an action of the group $\mathfrak{S}_{n+r}$ on the algebraic stacks $\mathscr{M}_{g,n+r} \subseteq \overline{\mathscr{M}}_{g,n+r}$ that arises from the permutations of $n+r$ marked points. We shall write

$$\mathscr{M}_{g,n+[r]} \stackrel{\text{def}}{=} [\mathscr{M}_{g,n+r}/\mathfrak{S}_{n+r,r}] \subseteq \overline{\mathscr{M}}_{g,n+[r]} \stackrel{\text{def}}{=} [\overline{\mathscr{M}}_{g,n+r}/\mathfrak{S}_{n+r,r}]$$

for the stack-theoretic quotients of the algebraic stacks $\mathscr{M}_{g,n+r} \subseteq \overline{\mathscr{M}}_{g,n+r}$ by the actions of the subgroup $\mathfrak{S}_{n+r,r} \subseteq \mathfrak{S}_{n+r}$ of $\mathfrak{S}_{n+r}$, respectively, and

$$\mathscr{D}_{g,n+[r]} \stackrel{\text{def}}{=} (\overline{\mathscr{M}}_{g,n+[r]} \setminus \mathscr{M}_{g,n+[r]})_{\text{red}} \subseteq \overline{\mathscr{M}}_{g,n+[r]}$$

for the reduced closed substack of $\overline{\mathscr{M}}_{g,n+[r]}$ determined by the complement of $\mathscr{M}_{g,n+[r]}$ in $\overline{\mathscr{M}}_{g,n+[r]}$.

(iii) We shall write

$$\mathscr{M}_{g,[r]} \stackrel{\text{def}}{=} \mathscr{M}_{g,0+[r]} \subseteq \overline{\mathscr{M}}_{g,[r]} \stackrel{\text{def}}{=} \overline{\mathscr{M}}_{g,0+[r]} \supseteq \mathscr{D}_{g,[r]} \stackrel{\text{def}}{=} \mathscr{D}_{g,0+[r]}.$$

(iv) Let I be a subset of $\{1, \ldots, n\}$. Then we shall write

$$\mathrm{pr}_I^{\mathscr{M}} : \overline{\mathscr{M}}_{g,n+[r]} \longrightarrow \overline{\mathscr{M}}_{g,n-\#I+[r]}$$

for the functor obtained by forgetting the marked points labeled by the elements of I.

Remark 2.3.1. One verifies immediately from the various definitions involved that the algebraic stack $\mathscr{M}_{g,[r]}$ may be naturally identified with the *moduli stack of hyperbolic curves of type* (g, r) over $\mathbb{Z}$.

Definition 2.4. Let S be a scheme, and let $X^+ = (X, D)$ be a hyperbolic curve of type (g, r) over S. Then we shall say that X^+ is *split* if the finite étale covering $D \to X \to S$ of S is trivial, or, alternatively, the classifying morphism $S \to \mathscr{M}_{g,[r]}$ of the hyperbolic curve X^+ [cf. Remark 2.3.1] factors through the natural finite étale covering $\mathscr{M}_{g,r} \to \mathscr{M}_{g,[r]}$.

Remark 2.4.1. Note that it is immediate that, for every hyperbolic curve (X, D) over a connected scheme S, there exists a connected finite étale covering $T \to S$ of S such that the hyperbolic curve $(X \times_S T, D \times_S T)$ over T is split.

Lemma 2.5. *The following assertions hold:*

(i) *Let n be a nonnegative integer. Then the diagram of stacks*

$$\begin{array}{ccc} \overline{\mathscr{M}}_{g,n+r} & \xrightarrow{\mathrm{pr}^{\mathscr{M}}_{\{1,\dots,n\}}} & \overline{\mathscr{M}}_{g,r} \\ \downarrow & & \downarrow \\ \overline{\mathscr{M}}_{g,n+[r]} & \xrightarrow[\mathrm{pr}^{\mathscr{M}}_{\{1,\dots,n\}}]{} & \overline{\mathscr{M}}_{g,[r]} \end{array}$$

— where the vertical arrows are the natural finite étale Galois coverings — is (1-)*cartesian.*

(ii) *Write r_0 for* 3 *(respectively,* 1*;* 0*) if $g = 0$ (respectively, $= 1$; $\geqslant 2$). [Thus, one verifies easily that $r_0 \leqslant r$.] Let σ be an element of $\mathfrak{S}_r$, and let $I \subseteq \{1, \dots, r\}$ be a subset of cardinality $r - r_0$. Then the diagram of stacks*

$$\begin{array}{ccc} \overline{\mathscr{M}}_{g,r} & \longrightarrow & \overline{\mathscr{M}}_{g,r} \\ & {\scriptstyle \mathrm{pr}^{\mathscr{M}}_I} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}^{\mathscr{M}}_I} & \\ & \overline{\mathscr{M}}_{g,r_0} & \end{array}$$

— where the upper horizontal arrow is the action of σ — is (1-)*commutative.*

Proof. Assertion (i) is immediate. Next, we verify assertion (ii). If $g \notin \{0, 1\}$, then this assertion is immediate. If $g = 0$, then this assertion follows immediately from the well-known fact that the natural morphism $\overline{\mathscr{M}}_{g,r_0} \to \mathrm{Spec}(\mathbb{Z})$ is an isomorphism. Suppose that $g = 1$. Let M be a scheme, $M \to \overline{\mathscr{M}}_{g,r_0}$ an étale surjective morphism [cf. [14, Theorem 2.7]], and $\overline{\eta} \to M$ a geometric generic point of M. Then since M is smooth and of finite type over $\mathbb{Z}$ [cf. [14, Theorem 2.7]], if one writes $M_{\overline{\eta}} \subseteq M$ for the connected component of M that contains the image of the morphism $\overline{\eta} \to M$, then the resulting morphism $\overline{\eta} \to M_{\overline{\eta}}$ is schematically dense. Thus, one verifies immediately that, to verify assertion (ii) in the case where $g = 1$, it suffices to verify that the diagram of stacks

$$\begin{array}{ccccc} \overline{\eta} & \longrightarrow & \overline{\mathscr{M}}_{g,r} & \longrightarrow & \overline{\mathscr{M}}_{g,r} \\ & \searrow & & & \downarrow {\scriptstyle \mathrm{pr}^{\mathscr{M}}_I} \\ & & \overline{\mathscr{M}}_{g,r} & \xrightarrow{\mathrm{pr}^{\mathscr{M}}_I} & \overline{\mathscr{M}}_{g,r_0} \end{array}$$

— where the arrows $\overline{\eta} \to \overline{\mathscr{M}}_{g,r}$ are the natural morphisms, and the right-hand upper horizontal arrow is the action of σ — is (1-)commutative. On the other hand, this assertion follows from the well-known fact that if (X, D) is a hyperbolic curve of type $(1, 1)$ over $\overline{\eta}$, then X admits a group scheme structure over $\overline{\eta}$, which implies that the natural action of the group of automorphisms of the scheme X over $\overline{\eta}$ on the set of closed points of X is transitive. This completes the proof of assertion (ii), hence also of Lemma 2.5. □

Definition 2.6. Let n be a nonnegative integer, S a scheme, and $X^+ = (X, D)$ a hyperbolic curve of type (g, r) over S. Then we shall write

$$X_{(n)} \stackrel{\text{def}}{=} \overline{\mathscr{M}}_{g,n+[r]} \times_{\overline{\mathscr{M}}_{g,[r]}} S$$

for the scheme over S obtained by forming the fiber product of the [representable — cf. [14, Corollary 2.6]] functor $\mathrm{pr}^{\mathscr{M}}_{\{1,\dots,n\}} \colon \overline{\mathscr{M}}_{g,n+[r]} \to \overline{\mathscr{M}}_{g,[r]}$ and the classifying morphism $S \to \overline{\mathscr{M}}_{g,[r]}$ of the hyperbolic curve X^+ [cf. Remark 2.3.1],

$$U^X_{(n)} \subseteq X_{(n)}, \qquad D^X_{(n)} \subseteq X_{(n)}$$

for the open, closed subschemes of $X_{(n)}$ determined by the open, closed substacks $\mathscr{M}_{g,n+[r]}$, $\mathscr{D}_{g,n+[r]} \subseteq \overline{\mathscr{M}}_{g,n+[r]}$ of $\overline{\mathscr{M}}_{g,n+[r]}$, respectively, and

$$X^+_{(n)} \stackrel{\text{def}}{=} (X_{(n)}, D^X_{(n)})$$

for the pair consisting of $X_{(n)}$ and $D^X_{(n)}$. We shall refer to $U^X_{(n)}$, $X_{(n)}$ as the *n-th configuration space* [cf. Remark 2.6.1, (i), below], *n-th compactified configuration space* of X^+, respectively.

Remark 2.6.1. In the situation of Definition 2.6:

(i) One verifies easily from the various definitions involved that the scheme $U^X_{(n)}$ may be naturally identified with the n-th configuration space of the curve $X \setminus D \subseteq X$ [cf., e.g., [21, Definition 2.1, (i)]].

(ii) One also verifies easily from the various definitions involved that the scheme $X_{(n)}$ may be naturally identified with the underlying scheme of the n-th log configuration space of the curve $X \setminus D \subseteq X$ [cf., e.g., [4, Definition 1], [21, Definition 2.1, (i)]].

Definition 2.7. Let n be a nonnegative integer, S a scheme, and $X^+ = (X, D)$ a hyperbolic curve of type (g, r) over S.

(i) We shall write

$$\varepsilon_{X^+}$$

for 3 (respectively, 1; 0) if the equality $(g, r) = (0, 3)$ holds, and X^+ is split (respectively, if the equality $(g, r) = (1, 1)$ holds; if either $(g, r) \notin \{(0, 3), (1, 1)\}$, or X^+ is not split).

(ii) Let I be a subset of $\{1, \dots, n\}$. Then we shall write

$$\mathrm{pr}_I \colon X_{(n)} \longrightarrow X_{(n-\#I)}$$

for the morphism over k determined by the functor $\mathrm{pr}^{\mathscr{M}}_I \colon \overline{\mathscr{M}}_{g,n+[r]} \to \overline{\mathscr{M}}_{g,n-\#I+[r]}$.

(iii) Let I be a subset of $\{1, \dots, n+\varepsilon_{X^+}\}$ of cardinality $\leqslant n$ such that $I \not\subseteq \{1, \dots, n\}$ [which thus implies that $r = \varepsilon_{X^+}$, and that the hyperbolic curve X^+ is split]. Fix a lifting $S \to \mathscr{M}_{g,\varepsilon_{X^+}}$ of the classifying morphism $S \to \mathscr{M}_{g,[\varepsilon_{X^+}]}$ of the hyperbolic curve X^+. Observe that it follows from Lemma 2.5, (i), that this lifting naturally determines an isomorphism over S

$$\overline{\mathscr{M}}_{g,n+\varepsilon_{X^+}} \times_{\overline{\mathscr{M}}_{g,\varepsilon_{X^+}}} S \xrightarrow{\sim} X_{(n)}.$$

Then we shall write

$$\mathrm{pr}_I \colon X_{(n)} \longrightarrow X_{(n-\#I)}$$

for the morphism over k determined by [cf. Lemma 2.5, (ii)] the functor $\mathrm{pr}_I^{\mathscr{M}} \colon \overline{\mathscr{M}}_{g,n+\varepsilon_{X^+}} \to \overline{\mathscr{M}}_{g,n+\varepsilon_{X^+}-\#I}$.

(iv) Suppose that $r \neq \varepsilon_{X^+}$. Then it is immediate that the action of $\mathfrak{S}_{n+r}$ on $\overline{\mathscr{M}}_{g,n+r}$ determines an action of $\mathfrak{S}_n$ on $X_{(n)}$. We shall refer to an automorphism of $X_{(n)}$ that arises from this action as a *modular symmetry automorphism* of $X_{(n)}$.

(v) Suppose that $r = \varepsilon_{X^+}$ [which thus implies that the hyperbolic curve X^+ is split]. Fix a lifting $S \to \mathscr{M}_{g,\varepsilon_{X^+}}$ of the classifying morphism $S \to \mathscr{M}_{g,[\varepsilon_{X^+}]}$ of the hyperbolic curve X^+. Observe that it follows from Lemma 2.5, (i), that this lifting naturally determines an isomorphism over S

$$\overline{\mathscr{M}}_{g,n+\varepsilon_{X^+}} \times_{\overline{\mathscr{M}}_{g,\varepsilon_{X^+}}} S \xrightarrow{\sim} X_{(n)}.$$

Then it follows from Lemma 2.5, (ii), that the action of $\mathfrak{S}_{n+\varepsilon_{X^+}}$ on $\overline{\mathscr{M}}_{g,n+\varepsilon_{X^+}}$ determines an action of $\mathfrak{S}_{n+\varepsilon_{X^+}}$ on $X_{(n)}$. We shall refer to an automorphism of $X_{(n)}$ that arises from this action as a *modular symmetry automorphism* of $X_{(n)}$.

Remark 2.7.1. Suppose that we are in the situation of Definition 2.7. Note that one main motivation of to define the integer ε_{X^+} [cf. Definition 2.7, (i)], as well as to consider a lifting $S \to \mathscr{M}_{g,\varepsilon_{X^+}}$ of the classifying morphism $S \to \mathscr{M}_{g,[\varepsilon_{X^+}]}$ [cf. Definition 2.7, (iii)], is to define the notion of a generalized fiber subgroup [cf. Definition 3.4 below].

Lemma 2.8. *Let n be a nonnegative integer, k a field, $X^+ = (X, D)$ a hyperbolic curve of type (g, r) over k, i an element of $\{1, \ldots, n + \varepsilon_{X^+} + 1\}$, and $Z_{n+1} \to U^X_{(n+1)}$ a connected finite étale covering tamely ramified along $D^X_{(n+1)}$. Let us fix a lifting $\mathrm{Spec}(k) \to \mathscr{M}_{g,r}$ of the classifying morphism $\mathrm{Spec}(k) \to \mathscr{M}_{g,[r]}$ of the hyperbolic curve X^+ whenever the hyperbolic curve X^+ is split. Write K for the algebraic closure of k in the function field of Z_{n+1} and $\overline{Z}_{n+1} \to X_{(n+1)}$ (respectively, $\overline{Z}_n \to X_{(n)}$) for the normalization of $X_{(n+1)}$ (respectively, $X_{(n)}$) in Z_{n+1}, i.e., relative to the given covering $Z_{n+1} \to U^X_{(n+1)}$ (respectively, to the composite of the given covering $Z_{n+1} \to U^X_{(n+1)}$ with the morphism $\mathrm{pr}_{\{i\}} \colon U^X_{(n+1)} \to U^X_{(n)}$). Write, moreover, E_n, E_{n+1} for the reduced closed subschemes of $\overline{Z}_n$, $\overline{Z}_{n+1}$ whose underlying closed subsets are given by the inverse images of $D^X_{(n)}$, $D^X_{(n+1)}$, respectively, and $Z_n \stackrel{\mathrm{def}}{=} \overline{Z}_n \setminus E_n$. Then the following assertions hold:*

(i) *Each of the pairs $Z_n^+ \stackrel{\mathrm{def}}{=} (\overline{Z}_n, E_n)$, $Z_{n+1}^+ \stackrel{\mathrm{def}}{=} (\overline{Z}_{n+1}, E_{n+1})$ is a good pair over K.*

(ii) *The morphism $\overline{Z}_{n+1} \to \overline{Z}_n$ induced by the morphism $\mathrm{pr}_{\{i\}} \colon X_{(n+1)} \to X_{(n)}$ is good with respect to (E_{n+1}, E_n) [cf. (i)].*

(iii) *Let $\overline{z} \to Z_n$ be a geometric point of Z_n. Write $(Z_{n+1})_{\overline{z}}$ for the geometric fiber at $\overline{z} \to Z_n$ of the morphism $\overline{Z}_{n+1} \to \overline{Z}_n$ induced by the morphism $\mathrm{pr}_{\{i\}} \colon X_{(n+1)} \to X_{(n)}$ and $\underline{\pi_1^{\text{ét}}}((Z_{n+1})_{\overline{z}}) \stackrel{\mathrm{def}}{=} \pi_1^{\text{ét}}((Z_{n+1})_{\overline{z}})$ (respectively, $\underline{\pi_1^{\text{ét}}}((Z_{n+1})_{\overline{z}})$ for the maximal pro-prime-to-$\mathrm{char}(k)$ quotient of $\pi_1^{\text{ét}}((Z_{n+1})_{\overline{z}})$) [cf. (ii); conditions (1), (3) of Definition 1.3] whenever the field k is of characteristic zero (respectively, of positive characteristic). Then the natural morphisms $(Z_{n+1})_{\overline{z}} \to \overline{Z}_{n+1} \to \overline{Z}_n$ determine an exact sequence of topological groups*

$$1 \longrightarrow \underline{\pi_1^{\text{ét}}}\big((Z_{n+1})_{\overline{z}}\big) \longrightarrow \underline{\Pi}^{\mathrm{tame}}_{Z^+_{n+1}/Z^+_n} \longrightarrow \pi_1^{\mathrm{tame}}(Z_n^+) \longrightarrow 1$$

[*cf.* (i); (ii); *Definition* 1.1, (iii); *Definition* 1.6, (ii)].

Proof. First, observe that it follows immediately from [14, Theorem 2.7] that

(a) each of the pairs $X^+_{(n)} = (X_{(n)}, D^X_{(n)})$, $X^+_{(n+1)} = (X_{(n+1)}, D^X_{(n+1)})$ is a good pair over k.

Next, we verify that

(b) the morphism $\mathrm{pr}_{\{i\}}\colon X_{(n+1)} \to X_{(n)}$ is good with respect to $(D^X_{(n+1)}, D^X_{(n)})$ [cf. (a)].

To this end, let us observe that it follows from the various definitions involved that the morphism $\mathrm{pr}_{\{i\}}\colon X_{(n+1)} \to X_{(n)}$ satisfies conditions (1), (2), (3), (4) of Definition 1.3. Moreover, it follows immediately from [14, Theorem 2.7], together with [13, Theorem 3.5], that the morphism $\mathrm{pr}_{\{i\}}\colon X_{(n+1)} \to X_{(n)}$ satisfies condition (5) of Definition 1.3. This completes the proof of the fact that the morphism $\mathrm{pr}_{\{i\}}\colon X_{(n+1)} \to X_{(n)}$ is good with respect to $(D^X_{(n+1)}, D^X_{(n)})$.

Next, write $\overline{Z}^{\log}_{n+1}$, $\overline{Z}^{\log}_n$, $X^{\log}_{(n)}$ for the [necessarily fs] log schemes obtained by equipping $\overline{Z}_{n+1}$, $\overline{Z}_n$, $X_{(n)}$ with the log structures determined by E_{n+1}, E_n, $D^X_{(n)}$, respectively. Then it follows immediately from Lemma 1.5, (iv), together with (b), that

(c) the factorization $\overline{Z}^{\log}_{n+1} \to \overline{Z}^{\log}_n \to X^{\log}_{(n)}$ is the log Stein factorization of the natural morphism $\overline{Z}^{\log}_{n+1} \to X^{\log}_{(n)}$ [i.e., determined by the composite of the given covering $Z_{n+1} \to U^X_{(n+1)}$ with the morphism $\mathrm{pr}_{\{i\}}\colon U^X_{(n+1)} \to U^X_{(n)}$].

Next, we verify assertion (i). It follows from Lemma 1.2, (ii), together with (a), that the pair $Z^+_{n+1} = (\overline{Z}_{n+1}, E_{n+1})$ is a good pair over k. Next, it follows from Lemma 1.5, (iii), together with (b) and (c), that the pair $Z^+_n = (\overline{Z}_n, E_n)$ is a good pair over k. This completes the proof of assertion (i). Moreover, assertion (ii) follows from Lemma 1.5, (v), together with (b) and (c). This completes the proof of assertion (ii).

Finally, we verify assertion (iii). Observe that it follows from Corollary 1.14, together with assertion (ii), that, to verify assertion (iii), it suffices to verify that the group $\underline{\pi_1^{\text{ét}}}((Z_{n+1})_{\overline{z}})$ is center-free. On the other hand, since [one verifies easily that] the geometric fiber $(Z_{n+1})_{\overline{z}}$ determines a hyperbolic curve over $\overline{z}$, this assertion is well-known [cf., e.g., [30, Corollary 1.4, (i), (ii)], [30, Proposition 1.11]]. This completes the proof of assertion (iii), hence also of Lemma 2.8. □

Lemma 2.9. *Let n be a nonnegative integer, k a separably closed field, $X^+ = (X, D)$ a hyperbolic curve of type (g, r) over k, i an element of $\{1, \ldots, n + \varepsilon_{X^+} + 1\}$, and $\overline{x} \to U^X_{(n)}$ a geometric point of $U^X_{(n)}$. Let us fix a lifting $\mathrm{Spec}(k) \to \mathscr{M}_{g,r}$ of the classifying morphism $\mathrm{Spec}(k) \to \mathscr{M}_{g,[r]}$ of the hyperbolic curve X^+ whenever the hyperbolic curve X^+ is split. Write $(D^X_{(n+1)})_{\overline{x}} \subseteq (X_{(n+1)})_{\overline{x}} \supseteq (U^X_{(n+1)})_{\overline{x}}$ for the respective geometric fibers at $\overline{x} \to U^X_{(n-1)}$ of $D^X_{(n)} \subseteq X_{(n)} \supseteq U^X_{(n)}$ with respect to the morphism $\mathrm{pr}_{\{i\}}\colon X_{(n+1)} \to X_{(n)}$, and $(X_{(n+1)})^+_{\overline{x}} \stackrel{\text{def}}{=} ((X_{(n+1)})_{\overline{x}}, (D^X_{(n+1)})_{\overline{x}})$. Thus, the morphism $\mathrm{pr}_{\{i\}}\colon X_{(n+1)} \to X_{(n)}$ and the natural morphism $(X_{(n+1)})_{\overline{x}} \to X_{(n+1)}$ determine a diagram of topological groups*

$$\begin{array}{ccccccccc} & & \pi_1^{\text{tame}}\big((X_{(n+1)})^+_{\overline{x}}\big) & & & & & & \\ & & \downarrow & & & & & & \\ 1 & \longrightarrow & \Delta^{\text{tame}}_{X^+_{(n+1)}/X^+_{(n)}} & \longrightarrow & \pi_1^{\text{tame}}(X^+_{(n+1)}) & \xrightarrow{\pi_1^{\text{tame}}(\mathrm{pr}_{\{i\}})} & \pi_1^{\text{tame}}(X^+_{(n)}) & \longrightarrow & 1 \end{array}$$

[cf. Definition 1.1, (iii); Lemma 1.4; Definition 1.6, (ii); Lemma 2.8, (i), (ii)] — where the lower sequence is exact [cf. Lemma 1.8; Lemma 2.8, (i), (ii)], and the left-hand vertical arrow is surjective [cf. Lemma 1.7; Lemma 2.8, (i), (ii)]. Then the following assertions hold:

(i) *The topological group $\Delta^{\mathrm{tame}}_{X^+_{(n+1)}/X^+_{(n)}}$ is topologically finitely generated.*

(ii) *Let $H \subseteq \Delta^{\mathrm{tame}}_{X^+_{(n+1)}/X^+_{(n)}}$ be an open subgroup of $\Delta^{\mathrm{tame}}_{X^+_{(n+1)}/X^+_{(n)}}$. Write $V \to (U^X_{(n+1)})_{\overline{x}}$ for the connected finite étale covering of $(U^X_{(n+1)})_{\overline{x}}$ tamely ramified along $(D^X_{(n+1)})_{\overline{x}}$ that corresponds to the open subgroup of $\pi_1^{\mathrm{tame}}((X_{(n+1)})^+_{\overline{x}})$ obtained by forming the inverse image of $H \subseteq \Delta^{\mathrm{tame}}_{X^+_{(n+1)}/X^+_{(n)}}$ by the left-hand vertical arrow $\pi_1^{\mathrm{tame}}((X_{(n+1)})^+_{\overline{x}}) \twoheadrightarrow \Delta^{\mathrm{tame}}_{X^+_{(n+1)}/X^+_{(n)}}$ of the above diagram. Thus, we have a natural continuous homomorphism $\pi_1^{\text{ét}}(V) \to H$. Then this continuous homomorphism $\pi_1^{\text{ét}}(V) \to H$ is a continuous isomorphism whenever the field k is of characteristic zero. Moreover, if the field k is of positive characteristic, then this continuous homomorphism $\pi_1^{\text{ét}}(V) \to H$ induces a continuous isomorphism of the maximal pro-prime-to-$\mathrm{char}(k)$ quotient of $\pi_1^{\text{ét}}(V)$ with the maximal pro-prime-to-$\mathrm{char}(k)$ quotient of H.*

(iii) *Let m be a positive integer, and let $H \subseteq \Delta^{\mathrm{tame}}_{X^+_{(n+1)}/X^+_{(n)}}$ be an open subgroup $\Delta^{\mathrm{tame}}_{X^+_{(n+1)}/X^+_{(n)}}$ of index $\geqslant m$. Write $V \to (U^X_{(n+1)})_{\overline{x}}$ for the connected finite étale covering of $(U^X_{(n+1)})_{\overline{x}}$ tamely ramified along $(D^X_{(n+1)})_{\overline{x}}$ that corresponds to the open subgroup of $\pi_1^{\mathrm{tame}}((X_{(n+1)})^+_{\overline{x}})$ obtained by forming the inverse image of $H \subseteq \Delta^{\mathrm{tame}}_{X^+_{(n+1)}/X^+_{(n)}}$ by the left-hand vertical arrow $\pi_1^{\mathrm{tame}}((X_{(n+1)})^+_{\overline{x}}) \twoheadrightarrow \Delta^{\mathrm{tame}}_{X^+_{(n+1)}/X^+_{(n)}}$ of the above diagram, $Y \to (X_{(n+1)})_{\overline{x}}$ for the normalization of $(X_{(n+1)})_{\overline{x}}$ in V, and $E \overset{\mathrm{def}}{=} (Y \setminus V)_{\mathrm{red}} \subseteq Y$ for the reduced closed subscheme of Y whose underlying closed subset is given by $Y \setminus V$. [Note that one verifies easily that the pair (Y, E) forms a hyperbolic curve over k.] Write, moreover, (g_H, r_H) for the pair of nonnegative integers so that (Y, E) is a hyperbolic curve of type (g_H, r_H) over k. Then the inequality $2g_H - 2 + r_H \geqslant m$ holds.*

(iv) *Consider the sequence of topological groups*

$$1 \longrightarrow (\Delta^{\mathrm{tame}}_{X^+_{(n+1)}/X^+_{(n)}})^{\{l\}} \longrightarrow \pi_1^{\mathrm{tame}}(X^+_{(n+1)})^{\{l\}} \longrightarrow \pi_1^{\mathrm{tame}}(X^+_{(n)})^{\{l\}} \longrightarrow 1$$

— where, for a profinite group "$(-)$", we write "$(-)^{\{l\}}$" for the maximal pro-l quotient of "$(-)$" — determined by the above diagram of topological groups. Then this sequence is exact.

Proof. First, we verify assertion (i). Since the left-hand vertical arrow $\pi_1^{\mathrm{tame}}((X_{(n+1)})^+_{\overline{x}}) \twoheadrightarrow \Delta^{\mathrm{tame}}_{X^+_{(n+1)}/X^+_{(n)}}$ of the diagram of the statement of the present Lemma 2.9 is surjective, to verify assertion (i), it suffices to verify that the topological group $\pi_1^{\mathrm{tame}}((X_{(n+1)})^+_{\overline{x}})$ is topologically finitely generated. On the other hand, this assertion follows from [30, Proposition 1.1, (i), (ii)]. This completes the proof of assertion (i).

Next, we verify assertion (ii). Observe that it is well-known [cf., e.g., [3, Lemma 1.2.5, (b)]] that there exists an open subgroup $\widetilde{H} \subseteq \pi_1^{\mathrm{tame}}(X^+_{(n+1)})$ of $\pi_1^{\mathrm{tame}}(X^+_{(n+1)})$ such that the equality $H = \widetilde{H} \cap \Delta^{\mathrm{tame}}_{X^+_{(n+1)}/X^+_{(n)}}$ holds. Thus, one concludes the desired conclusion by applying Lemma 2.8, (iii), in the case where the "$Z_{n+1} \to U^X_{(n+1)}$" of Lemma 2.8 to be the connected finite étale covering of $U^X_{(n+1)}$ tamely ramified along $D^X_{(n+1)}$ that corresponds to the open subgroup $\widetilde{H} \subseteq \pi_1^{\mathrm{tame}}(X^+_{(n+1)})$ of $\pi_1^{\mathrm{tame}}(X^+_{(n+1)})$. This completes the proof of assertion (ii). Assertion (iii) is an immediate consequence of Hurwitz's formula.

Finally, we verify assertion (iv). Observe that the morphism $\mathrm{pr}_{\{i\}} \colon X_{(n+1)} \to X_{(n)}$ and the natural morphism $(X_{(n+1)})_{\overline{x}} \to X_{(n+1)}$ determine a diagram of topological groups

$$\begin{array}{ccccccccc} & & \pi_1^{\text{ét}}\big((U^X_{(n+1)})_{\overline{x}}\big) & & & & & & \\ & & \downarrow & & & & & & \\ 1 & \longrightarrow & \Delta_{U^X_{(n+1)}/U^X_{(n)}} & \longrightarrow & \pi_1^{\text{ét}}(U^X_{(n+1)}) & \xrightarrow{\pi_1^{\text{ét}}(\mathrm{pr}_{\{i\}})} & \pi_1^{\text{ét}}(U^X_{(n)}) & \longrightarrow & 1 \end{array}$$

[cf. Definition 1.1, (iii); Definition 1.6, (i); Lemma 2.8, (i), (ii)], which determines a sequence of topological groups

$$1 \longrightarrow \pi_1^{\text{ét}}\big((U^X_{(n+1)})_{\overline{x}}\big)^{\{l\}} \longrightarrow \pi_1^{\text{ét}}(U^X_{(n+1)})^{\{l\}} \xrightarrow{\pi_1^{\text{ét}}(\mathrm{pr}_{\{i\}})} \pi_1^{\text{ét}}(U^X_{(n)})^{\{l\}} \longrightarrow 1$$

— where, for a profinite group "$(-)$", we write "$(-)^{\{l\}}$" for the maximal pro-l quotient of "$(-)$". Then it follows from [21, Proposition 2.2, (i)] that this sequence is exact. Thus, assertion (iv) follows immediately from the [easily verified] fact that each of the natural continuous surjective homomorphisms $\pi_1^{\text{ét}}((U^X_{(n+1)})_{\overline{x}})^{\{l\}} \twoheadrightarrow \pi_1^{\mathrm{tame}}((X_{(n+1)})^+_{\overline{x}})^{\{l\}}$, $\pi_1^{\text{ét}}(U^X_{(n+1)})^{\{l\}} \twoheadrightarrow \pi_1^{\mathrm{tame}}(X^+_{(n+1)})^{\{l\}}$, $\pi_1^{\text{ét}}(U^X_{(n)})^{\{l\}} \twoheadrightarrow \pi_1^{\mathrm{tame}}(X^+_{(n)})^{\{l\}}$ is a continuous isomorphism. This completes the proof of assertion (iv), hence also of Lemma 2.9. □

3. Generalities on generalized fiber subgroups

In the present §3, we introduce and discuss generalized fiber subgroups of the tame fundamental groups of the compactified configuration spaces of hyperbolic curves [cf. Definition 3.4 below]. In particular, we verify some group-theoretic properties of generalized fiber subgroups [cf. Theorem 3.7 below].

Definition 3.1. Let G be a profinite group, and let Σ be a set of prime numbers.

(i) We shall write

$$G^{\Sigma}$$

for the maximal pro-Σ quotient of G.

(ii) Let $N \subseteq G$ be a normal open subgroup of G. Then we shall define the *almost pro-Σ-maximal quotient* of G associated to N [cf. [21, Definition 1.1, (iii)]] to be the quotient $G/\mathrm{Ker}(N \twoheadrightarrow N^{\Sigma})$ of G by the kernel of the natural continuous surjective homomorphism $N \twoheadrightarrow N^{\Sigma}$. [Observe that since N is normal in G, and $\mathrm{Ker}(N \twoheadrightarrow N^{\Sigma})$ is characteristic in N, one verifies easily that $\mathrm{Ker}(N \twoheadrightarrow N^{\Sigma})$ is normal in G.] We shall define an *almost pro-Σ-maximal quotient* of G [cf. [21, Definition 1.1, (iii)]] to be the almost pro-Σ-maximal quotient of G associated to some normal open subgroup of G.

Definition 3.2. Let G be a profinite group.

(i) We shall say that G is *slim* [cf. [21, §0]] if the centralizer in G of every open subgroup of G is trivial.

(ii) We shall say that a subgroup $H \subseteq G$ of G is *subnormal* [cf. [15, Definition 1.1]] if there exist a positive integer m and a sequence $H = H_m \subseteq \ldots \subseteq H_2 \subseteq H_1 = G$ of subgroups of G such that H_i is normal in H_{i-1} for each $i \in \{2, \ldots, m\}$.

(iii) We shall say that G is *sn-internally indecomposable* [cf. [15, Definition 1.8, (iii)]] if the centralizer in G of every nontrivial subnormal subgroup of G is trivial.

(iv) We shall say that G is *strongly sn-internally indecomposable* [cf. [15, Definition 1.8, (iii)]] if every open subgroup of G is sn-internally indecomposable.

(v) We shall say that G is *sn-quasi-elastic* (respectively, *quasi-elastic*) [cf. [15, Definition 2.1, (i), (ii)]] if every nontrivial topologically finitely generated subnormal (respectively, normal) closed subgroup of G is open.

(vi) We shall say that G is *sn-elastic* [cf. [15, Definition 2.1, (ii)]] if every open subgroup of G is sn-quasi-elastic.

Lemma 3.3. *Let k be a separably closed field, and let Σ be a nonempty set of prime numbers invertible in k. Then every almost pro-Σ-maximal quotient of the étale fundamental group of a hyperbolic curve over k is topologically finitely generated, strongly sn-internally indecomposable, and sn-elastic.*

Proof. Let G be a topological group isomorphic to the étale fundamental group of a hyperbolic curve over k, and let $N \subseteq G$ be a normal open subgroup of G. Write Q for the almost pro-Σ-maximal quotient of G associated to N. Then it follows from [30, Proposition 1.1, (i), (ii)] that N^{Σ}, hence also Q, is topologically finitely generated. Moreover, it is well-known [cf., e.g., [11, Lemma 2.14, (i)]] that the continuous outer action of G/N on N^{Σ} is faithful — which thus implies that the image in G/N of the center of Q is trivial. In particular, since N^{Σ} is slim [cf., e.g., [30, Corollary 1.4, (i), (ii)], [30, Proposition 1.11]], it follows — by considering the natural exact sequence $1 \to N^{\Sigma} \to Q \to G/N \to 1$ — from the various definitions involved that Q is slim. Thus, it follows from [15, Proposition 1.12] (respectively, [15, Lemma 2.3]) that, to verify that Q is strongly sn-internally indecomposable (respectively, sn-elastic), it suffices to verify that N^{Σ} is strongly sn-internally indecomposable (respectively, sn-elastic). On the other hand, this assertion follows from [15, Theorem 3.13]. This completes the proof of Lemma 3.3. □

In the remainder of the present §3, let

- n be a positive integer,
- g, r nonnegative integers such that $2 - 2g - r < 0$,
- k a separably closed field,
- X^+ a hyperbolic curve of type (g, r) over k,
- Σ a set of prime numbers, and
- l a prime number contained in Σ invertible in k.

In particular, the hyperbolic curve X^+ is split. Let us fix a lifting $\mathrm{Spec}(k) \to \mathscr{M}_{g,r}$ of the classifying morphism $\mathrm{Spec}(k) \to \mathscr{M}_{g,[r]}$ of the hyperbolic curve X^+ [cf. Remark 2.3.1]. For each $i \in \{0, \ldots, n\}$, we shall write

$$\Pi_i \stackrel{\mathrm{def}}{=} \pi_1^{\mathrm{tame}}(X^+_{(i)})^{\Sigma}$$

[cf. Definition 1.1, (iii); Definition 2.6; Lemma 2.8, (i)].

Definition 3.4. Let I be a subset of $\{1, \ldots, n+\varepsilon_{X^+}\}$ of cardinality $\leqslant n$, and let i be an element of $\{0, \ldots, n\}$. Then we shall write

$$F_I \overset{\text{def}}{=} F_I(\Pi_n) \subseteq \Pi_n$$

for the kernel of the continuous outer [necessarily surjective — cf. Remark 3.4.1 below] homomorphism $\Pi_n \to \Pi_{n-\#I}$ induced by the morphism $\mathrm{pr}_I \colon X_{(n)} \to X_{(n-\#I)}$ [cf. Definition 2.7, (ii), (iii)] and refer to $F_I \subseteq \Pi_n$ as the *generalized fiber subgroup* of Π_n associated to I. We shall define a *generalized fiber subgroup* of Π_n to be the generalized fiber subgroup of Π_n associated to some subset of $\{1, \ldots, n+\varepsilon_{X^+}\}$ of cardinality $\leqslant n$. We shall say that a generalized fiber subgroup of Π_n is *of co-length* i if the subgroup is the generalized fiber subgroup of Π_n associated to some subset of $\{1, \ldots, n+\varepsilon_{X^+}\}$ of cardinality $n-i$. We shall write

$$\mathrm{GFS}_i(\Pi_n)$$

for the set of generalized fiber subgroups of Π_n of co-length i.

Remark 3.4.1. Note that, in the situation of Definition 3.4, it follows immediately from Lemma 2.8, (iii), that the continuous outer homomorphism $\Pi_n \to \Pi_{n-\#I}$ induced by the morphism $\mathrm{pr}_I \colon X_{(n)} \to X_{(n-\#I)}$ is surjective.

Lemma 3.5. *Let I be a subset of $\{1, \ldots, n+\varepsilon_{X^+}\}$ of cardinality $\leqslant n$. Write $F_I(l) \subseteq \Pi_n^{\{l\}}$ for the image in $\Pi_n^{\{l\}}$ of the generalized fiber subgroup $F_I \subseteq \Pi_n$ of Π_n associated to I. Then the following assertions hold:*

(i) *The subgroup $F_I(l) \subseteq \Pi_n^{\{l\}}$ is the generalized fiber subgroup of $\Pi_n^{\{l\}}$ [i.e., the "Π_n" in the case where we take the "Σ" to be $\{l\}$] associated to I.*

(ii) *The continuous surjective homomorphism $F_I^{\{l\}} \twoheadrightarrow F_I(l)$ induced by the natural continuous surjective homomorphism $F_I \twoheadrightarrow F_I(l)$ is an isomorphism.*

Proof. Assertion (i) follows from the well-known fact that the operation of taking the maximal pro-l quotient is right exact. Next, we verify assertion (ii) by induction on $\#I$. If $\#I = 0$, then the assertion (ii) is immediate. Suppose that $\#I > 0$, and that the induction hypothesis is in force. Let i be an element of I. Then one verifies immediately from assertion (i), together with the induction hypothesis, that, to verify assertion (ii), we may assume without loss of generality, by replacing $F_I \subseteq \Pi_n$ by $F_I/F_{I\setminus\{i\}} \subseteq \Pi_n/F_{I\setminus\{i\}} = \Pi_{n-\#I+1}$, that $\#I = 1$. On the other hand, assertion (ii) in the case where $\#I = 1$ is none other than Lemma 2.9, (iv). This completes the proof of assertion (ii), hence also of Lemma 3.5. ☐

Remark 3.5.1. In the situation of Lemma 3.5, suppose that $I \neq \emptyset$. Then one verifies immediately from Lemma 3.5, (i), (ii), together with [21, Proposition 2.2, (i)], that the maximal pro-l quotient $F_I^{\{l\}}$ may be naturally identified with the "Π_n" for a suitable hyperbolic curve over a separably closed field for which the "(n, g, r, Σ)" is given by $(\#I, g, r+n-\#I, \{l\})$.

Lemma 3.6. *Suppose that Σ is the set of all prime numbers. Let $F \subseteq \Pi_n$ be a generalized fiber subgroup of Π_n of co-length $n-1$, Σ' a set of prime numbers invertible in k, and $N \subseteq \Pi_n$ a normal open subgroup of Π_n. Then the almost pro-Σ'-maximal quotient of F associated to $N \cap F$ is isomorphic to an almost pro-Σ'-maximal quotient of the étale fundamental group of a hyperbolic curve over k.*

Proof. This assertion is a formal consequence of Lemma 2.9, (ii). ☐

Theorem 3.7. *Suppose that Σ is the set of all prime numbers. Let $F \subseteq \Pi_n$ be a generalized fiber subgroup of Π_n. Then the following assertions hold:*

(i) *If F is of co-length $n-1$, then the profinite group F is strongly sn-internally indecomposable and sn-elastic.*

(ii) *The profinite group F is topologically finitely generated and slim.*

Proof. First, we verify assertion (i). It follows from Lemma 3.3 and Lemma 3.6, together with [15, Proposition 1.14], that F is strongly sn-internally indecomposable. Next, observe that it follows immediately from a similar argument to the argument applied in the proof of [15, Lemma 2.8] that, to verify the sn-elasticity of F, it suffices to verify that, for each positive integer m, there exists a positive integer d_m such that every open subgroup $U \subseteq F$ of F of index $\geqslant d_m$ satisfies the following two conditions:

(1) There exists a set S of normal open subgroups of U such that $\bigcap_{N \in S} N = \{1\}$, and, moreover, for each $N \in S$, the almost pro-l-maximal quotient of U associated to N is sn-quasi-elastic.

(2) There is no open subgroup of $U^{\{l\}}$ topologically generated by m elements.

To this end, let us first observe that it follows immediately from Lemma 3.3 and Lemma 3.6 that every open subgroup of F satisfies condition (1). Moreover, it follows immediately from Lemma 2.9, (ii), (iii), and [30, Corollary 1.2] that, for each positive integer m, every open subgroup of F of index $\geqslant m$ satisfies condition (2). This completes the proof of the sn-elasticity of F, hence also of assertion (i).

Next, we verify assertion (ii). Let us first observe that one verifies easily that an extension of a profinite group that is topologically finitely generated (respectively, slim) by a profinite group that is topologically finitely generated (respectively, slim) is topologically finitely generated (respectively, slim). Moreover, one also verifies easily that, for each $J \subseteq I \subseteq \{1, \ldots, n + \varepsilon_{X^+}\}$ of cardinality $\leqslant n$ such that $\#I = \#J + 1$, the quotient $F_I/F_J \subseteq \Pi_n/F_J = \Pi_{n-\#J}$ is a generalized fiber subgroup of $\Pi_{n-\#J}$ of co-length $n - \#J - 1$. Thus, to verify assertion (ii), we may assume without loss of generality that F is of co-length $n-1$. Then it follows from Lemma 2.9, (i), that F is topologically finitely generated. Moreover, it follows from assertion (i) that F is strongly sn-internally indecomposable, hence also slim. This completes the proof of assertion (ii), hence also of Theorem 3.7. □

4. Reconstruction of generalized fiber subgroups

In the present §4, we establish a "group-theoretic reconstruction algorithm" of generalized fiber subgroups [and their invariants] [cf. Theorem 4.8 below and Corollary 4.9 below]. In the present §4, we shall apply the notational conventions introduced at the discussion preceding Definition 3.4.

Lemma 4.1. *Let G, H be profinite groups, $\phi \colon G \to H$ a continuous homomorphism, and p a prime number. Suppose that the following four conditions are satisfied:*

(1) *The image of ϕ is normal in H.*

(2) *The image of the composite of ϕ with the natural continuous surjective homomorphism $H \twoheadrightarrow H^{\{p\}}$ [cf. Definition 3.1, (i)] is not open in $H^{\{p\}}$.*

(3) *Every almost pro-p-maximal quotient of G is topologically finitely generated.*

(4) *There exists a set S of normal open subgroups of H such that $\bigcap_{N\in S} N = \{1\}$, and, moreover, for each $N \in S$, the almost pro-p-maximal quotient of H associated to N is quasi-elastic.*

Then the image of the homomorphism ϕ is trivial.

Proof. Let N be an element of S. Write $M \stackrel{\text{def}}{=} \phi^{-1}(N) \subseteq G$ and Q_G (respectively, Q_H) for the almost pro-p-maximal quotient of G (respectively, H) associated to M (respectively, N). Then it is immediate that the composite of ϕ with the natural continuous surjective homomorphism $H \twoheadrightarrow Q_H$ factors through the natural continuous surjective homomorphism $G \twoheadrightarrow Q_G$. Write $\phi_Q \colon Q_G \to Q_H$ for the resulting continuous homomorphism. Then it follows from conditions (1), (3) that the image of ϕ_Q is a topologically finitely generated normal closed subgroup of Q_H. In particular, it follows from condition (4) that the image of ϕ_Q is either trivial or open in Q_H. Thus, it follows from condition (2) that the image of ϕ_Q is trivial, which implies that the image of ϕ is contained in N. In particular, the image of ϕ is contained in $\bigcap_{N\in S} N = \{1\}$ [cf. condition (4)], as desired. This completes the proof of Lemma 4.1. □

Lemma 4.2. *Let I, J be subsets of $\{1, \ldots, n+\varepsilon_{X^+}\}$ of cardinality $\leqslant n$. Suppose that $J \subseteq I$, and that $\#I = \#J + 1$. Let $G \subseteq F_I$ be a normal closed subgroup of F_I [cf. Definition 3.4]. Suppose, moreover, that the following two conditions are satisfied:*

(1) *Every almost pro-l-maximal quotient of G is topologically finitely generated.*
(2) *Write $F_J(l) \subseteq F_I(l) \subseteq \Pi_n^{\{l\}}$ for the respective images of $F_J \subseteq F_I \subseteq \Pi_n$ in $\Pi_n^{\{l\}}$. Then the image of $G \subseteq F_I$ in $F_I(l)/F_J(l)$ is not open in $F_I(l)/F_J(l)$.*

Then the inclusion $G \subseteq F_J$ holds.

Proof. Observe that it follows from condition (1) that the profinite group G satisfies condition (3) of Lemma 4.1 [in the case where we take the "p" of Lemma 4.1 to be l]. Write $H \stackrel{\text{def}}{=} F_I/F_J$. Then since H is naturally isomorphic to the kernel of the natural continuous surjective homomorphism $\Pi_n/F_J \twoheadrightarrow \Pi_n/F_I$, it follows from Lemma 3.3 and Lemma 3.6 that the profinite group H satisfies condition (4) of Lemma 4.1 [in the case where we take the "p" of Lemma 4.1 to be l]. Moreover, since [it is immediate that] the natural continuous surjective homomorphism $F_I \twoheadrightarrow F_I(l)/F_J(l)$ factors through the natural continuous surjective homomorphism $F_I \twoheadrightarrow H^{\{l\}}$, it follows from condition (2) that the image of G in $H^{\{l\}}$ is not open. Thus, since [it is immediate that] the image of G in H is normal, one concludes from Lemma 4.1 [in the case where we take the "p" of Lemma 4.1 to be l] that the image of $G \subseteq F_I$ in H is trivial, which thus implies that $G \subseteq F_J$, as desired. This completes the proof of Lemma 4.2. □

Definition 4.3. Let Π be a profinite group, p a prime number, and S a set of normal closed subgroups of $\Pi^{\{p\}}$. Then we shall write

$$\mathscr{F}_p(\Pi, S)$$

for the set of normal closed subgroups $G \subseteq \Pi$ of Π that satisfy the following two conditions:

(1) Every almost pro-p-maximal quotient of G is topologically finitely generated.
(2) There exists an element $N \in S$ such that the image of G in $\Pi^{\{p\}}/N$ is not open in $\Pi^{\{p\}}/N$.

Moreover, we shall write

$$\overline{\mathscr{F}}_p(\Pi, S)$$

for the set of maximal elements [with respect to inclusion] of $\mathscr{F}_p(\Pi, S)$.

Lemma 4.4. *Let I be a subset of $\{1, \dots, n+\varepsilon_{X^+}\}$ of cardinality $\leqslant n$. Then the following assertions hold:*

(i) *Suppose that $\#I < n$. Let I' be a subset of $\{1, \dots, n+\varepsilon_{X^+}\}$ of cardinality $< n$. Then the inclusion $I' \subseteq I$ holds if and only if the inclusion $F_{I'} \subseteq F_I$ holds.*

(ii) *Note that, in this situation, it follows from Definition 3.4 and Remark 3.5.1 that the notation $\mathrm{GFS}_1(F_I^{\{l\}})$ makes sense. Let G be an element of $\mathscr{F}_l(F_I, \mathrm{GFS}_1(F_I^{\{l\}}))$. Then there exists a generalized fiber subgroup F' of Π_n of co-length $n - \#I + 1$ contained in F_I such that $G \subseteq F'$.*

(iii) *Every generalized fiber subgroup of Π_n of co-length $n-\#I+1$ contained in F_I is an element of $\mathscr{F}_l(F_I, \mathrm{GFS}_1(F_I^{\{l\}}))$.*

Proof. First, we verify assertion (i). Necessity is immediate. Next, we verify sufficiency. Assume that the inclusion $F_{I'} \subseteq F_I$ holds, but that the inclusion $I' \subseteq I$ does not hold. Observe that it follows from Lemma 3.5, (i), (ii), that, to obtain a contradiction, we may assume without loss of generality, by replacing Σ by $\{l\}$, that $\Sigma = \{l\}$. Next, observe that it is immediate that, to obtain a contradiction, we may assume without loss of generality, by replacing I' by a suitable subset of I', that I' is of cardinality one, which thus [cf. our assumption that $\#I < n$] implies that the union $I \cup I'$ is of cardinality $\leqslant n$. Thus, to obtain a contradiction, we may assume without loss of generality, by applying a suitable modular symmetry automorphism of $X_{(n)}$, that the union $I \cup I'$ is contained in $\{1, \dots, n\}$. In particular, it follows immediately from [21, Proposition 2.4, (iv)] [cf. also the above assumption that $\Sigma = \{l\}$] that the composite $F_{I'} \hookrightarrow F_{I\cup I'} \twoheadrightarrow F_{I\cup I'}/F_I$ is surjective. Thus, since [we have assumed that] the inclusion $F_{I'} \subseteq F_I$ holds, one concludes the equality $F_{I\cup I'} = F_I$, in contradiction to the nontriviality of the quotient $F_{I\cup I'}/F_I$, which may be derived from [30, Corollary 1.4, (i), (ii)] and [21, Proposition 2.2, (i)] [cf. also the above assumption that $I' \not\subseteq I$]. This completes the proof of assertion (i).

Assertion (ii) is a formal consequence of Lemma 4.2 [cf. also Lemma 3.5, (i), (ii)]. Next, we verify assertion (iii). First, observe that it follows from assertion (i) that, to verify assertion (iii), it suffices to verify that, for a subset $J \subseteq I$ such that $\#I = \#J + 1$, the subgroup F_J is an element of $\mathscr{F}_l(F_I, \mathrm{GFS}_1(F_I^{\{l\}}))$. To this end, observe that it follows from Theorem 3.7, (ii), that F_J satisfies condition (1) of Definition 4.3. Next, it is immediate that the image of $F_J \subseteq F_I$ in $F_I^{\{l\}}/F_J^{\{l\}}$ [cf. Lemma 3.5, (i), (ii)] is trivial, which thus implies [cf. [30, Proposition 1.1, (i), (ii)], [21, Proposition 2.2, (i)]] that F_J satisfies condition (2) of Definition 4.3. Thus, the subgroup F_J is an element of $\mathscr{F}_l(F_I, \mathrm{GFS}_1(F_I^{\{l\}}))$, as desired. This completes the proof of assertion (iii), hence also of Lemma 4.4. □

Lemma 4.5. *Let i be an element of $\{0, \dots, n-1\}$, and let F be a generalized fiber subgroup of Π_n of co-length i. Then the set of generalized fiber subgroups of Π_n of co-length $i+1$ contained in F coincides with the set $\overline{\mathscr{F}}_l(F, \mathrm{GFS}_1(F^{\{l\}}))$. In particular, the equalities*

$$\mathrm{GFS}_0(\Pi_n) = \{\Pi_n\}, \qquad \mathrm{GFS}_{i+1}(\Pi_n) = \bigcup_{F \in \mathrm{GFS}_i(\Pi_n)} \overline{\mathscr{F}}_l\big(F, \mathrm{GFS}_1(F^{\{l\}})\big)$$

hold.

Proof. This assertion follows immediately from Lemma 4.4, (i), (ii), (iii). □

Lemma 4.6. *Let p be a prime number. Consider the following four conditions:*

(1) *The inclusion $p \in \Sigma \setminus (\Sigma \cap \{\mathrm{char}(k)\})$ holds.*

(2) *The group* $\Pi_n^{\{p\}}$ *is nontrivial.*

(3) *The inclusion* $p \in \Sigma$ *holds.*

(4) *The topological group* Π_n *is not a pro-prime-to-*p *group.*

Then the implications

$$(1) \Longrightarrow (2) \Longrightarrow (3) \Longleftrightarrow (4)$$

hold.

Proof. Let us first observe that the implications (2) $\Rightarrow$ (3) $\Leftarrow$ (4) are immediate. Next, observe that, to verify the implications (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4), we may assume without loss of generality, by replacing Π_n by Π_1, that $n = 1$. On the other hand, the implications (1) $\Rightarrow$ (2) in the case where $n = 1$ follows from [30, Corollary 1.2]. Moreover, the implications (3) $\Rightarrow$ (4) in the case where $n = 1$ follows immediately from [30, Corollary 1.2] and [30, Lemma 1.9]. This completes the proof of Lemma 4.6. □

Definition 4.7. Let i be a nonnegative integer, and let G be a profinite group.

(i) We shall define the set of closed subgroups of G

$$\{*\}\text{-GFS}_1(G)$$

as follows:

- If G is a pro-p configuration space group [i.e., a profinite group isomorphic to the maximal pro-p quotient of the étale fundamental group of the j-th configuration space of a hyperbolic curve over a separably closed field of characteristic $\neq p$ for some positive integer j — cf. [21, Definition 2.3, (i)]] for some prime number p, then $\{*\}\text{-GFS}_1(G)$ is defined to be the set of generalized fiber subgroups of G of co-length one [cf. Definition A.3, (ii), (iii)].
- If G is not a pro-p configuration space group for every prime number p, then $\{*\}\text{-GFS}_1(G) \stackrel{\text{def}}{=} \emptyset$.

(ii) Let p be a prime number. Then we shall define the set of closed subgroups of G

$$\underline{\text{GFS}}_i(G, p)$$

as follows:

- If $G^{\{p\}}$ is trivial, then $\underline{\text{GFS}}_i(G, p) \stackrel{\text{def}}{=} \emptyset$.
- If $G^{\{p\}}$ is nontrivial, then $\underline{\text{GFS}}_0(G, p) \stackrel{\text{def}}{=} \{G\}$.
- If $G^{\{p\}}$ is nontrivial, then

$$\underline{\text{GFS}}_{i+1}(G, p) \stackrel{\text{def}}{=} \bigcup_{F \in \underline{\text{GFS}}_i(G,p)} \overline{\mathscr{F}}_p\big(F, \{*\}\text{-GFS}_1(F^{\{p\}})\big).$$

(iii) We shall say that a prime number p is $\underline{\text{GFS}}$-*trivial with respect to* G if $G^{\{p\}}$ is trivial, or, alternatively, the equality $\underline{\text{GFS}}_j(G, p) = \emptyset$ holds for every nonnegative integer j.

(iv) We shall say that a prime number p is $\underline{\text{GFS}}$-*special with respect to* G if the following condition is satisfied: Let q_1, q_2 be prime numbers such that $q_1 \neq p$, $q_2 \neq p$, and, moreover, neither q_1 nor q_2 is $\underline{\text{GFS}}$-trivial with respect to G. Then the equality $\underline{\text{GFS}}_j(G, q_1) = \underline{\text{GFS}}_j(G, q_2)$ holds for each nonnegative integer j.

(v) We shall say that a prime number p is *strictly* $\underline{\mathrm{GFS}}$*-special with respect to* G if p is $\underline{\mathrm{GFS}}$-special with respect to G, and, moreover, there are at least two distinct prime numbers q_1, q_2 such that $q_1 \neq p$, $q_2 \neq p$, and, moreover, neither q_1 nor q_2 is $\underline{\mathrm{GFS}}$-trivial with respect to G.

(vi) We shall define the set of closed subgroups of G

$$\underline{\mathrm{GFS}}_i(G)$$

as follows:

– If G is a pro-p configuration space group for some prime number p, then $\underline{\mathrm{GFS}}_i(G)$ is defined to be the set of generalized fiber subgroups of G of co-length i [cf. Definition A.3, (iv)].

– If G is not a pro-p configuration space group for every prime number p, and there is no prime number strictly $\underline{\mathrm{GFS}}$-special with respect to G, then $\underline{\mathrm{GFS}}_i(G) \stackrel{\mathrm{def}}{=} \emptyset$.

– If there exists a prime number p strictly $\underline{\mathrm{GFS}}$-special with respect to G [which thus implies that G is not a pro-p' configuration space group for every prime number p'], then $\underline{\mathrm{GFS}}_i(G) \stackrel{\mathrm{def}}{=} \underline{\mathrm{GFS}}_i(G, q)$, where q is a prime number such that $q \neq p$, and, moreover, q is not $\underline{\mathrm{GFS}}$-trivial with respect to G. [One verifies immediately from the various definitions involved that this "$\underline{\mathrm{GFS}}_i(G)$" does not depend on the choices of such prime numbers "p", "q".]

Theorem 4.8. *Let i be an element of $\{0, \ldots, n\}$. Suppose that the set Σ is nonempty, and that the inequality $3 \leqslant \#\Sigma$ holds whenever* $\mathrm{char}(k) \in \Sigma$. *Then the equality*

$$\mathrm{GFS}_i(\Pi_n) = \underline{\mathrm{GFS}}_i(\Pi_n)$$

holds.

Proof. This assertion is a formal consequence of Lemma 4.5, Theorem A.4, and the implications (1) ⇒ (2) ⇒ (3) of Lemma 4.6. □

Corollary 4.9. *For each $\square \in \{\dagger, \ddagger\}$, let*

– *${}^{\square}n$ be a positive integer,*
– *${}^{\square}g$, ${}^{\square}r$ nonnegative integers such that $2 - 2{}^{\square}g - {}^{\square}r < 0$,*
– *${}^{\square}k$ a separably closed field,*
– *${}^{\square}X^{+}$ a hyperbolic curve of type $({}^{\square}g, {}^{\square}r)$ over ${}^{\square}k$, and*
– *${}^{\square}\Sigma$ a set of prime numbers.*

Suppose that, for each $\square \in \{\dagger, \ddagger\}$, the set ${}^{\square}\Sigma$ is nonempty, and the inequality $3 \leqslant \#{}^{\square}\Sigma$ holds whenever $\mathrm{char}({}^{\square}k) \in {}^{\square}\Sigma$. *Let*

$$\alpha\colon \pi_1^{\mathrm{tame}}({}^{\dagger}X^{+}_{({}^{\dagger}n)})^{\dagger\Sigma} \xrightarrow{\sim} \pi_1^{\mathrm{tame}}({}^{\ddagger}X^{+}_{({}^{\ddagger}n)})^{\ddagger\Sigma}$$

[cf. *Definition* 1.1, (iii)*; Definition* 2.6*; Lemma* 2.8, (i)] *be a continuous isomorphism. Then the following assertions hold:*

(i) *The equalities*

$${}^{\dagger}\Sigma = {}^{\ddagger}\Sigma, \qquad {}^{\dagger}n = {}^{\ddagger}n$$

hold.

(ii) *Let i be an element of* $\{0, \ldots, {}^\dagger n = {}^\ddagger n\}$ [*cf.* (i)]. *Then the isomorphism α determines a bijective map*

$$\mathrm{GFS}_i\big(\pi_1^{\mathrm{tame}}({}^\dagger X^+_{({}^\dagger n)})^{{}^\dagger\Sigma}\big) \xrightarrow{\sim} \mathrm{GFS}_i\big(\pi_1^{\mathrm{tame}}({}^\ddagger X^+_{({}^\ddagger n)})^{{}^\ddagger\Sigma}\big).$$

(iii) *If, moreover, the inequality* ${}^\dagger n = {}^\ddagger n \geqslant 2$ [*cf.* (i)] *holds, then the equalities*

$${}^\dagger g = {}^\ddagger g, \qquad {}^\dagger r = {}^\ddagger r$$

hold.

(iv) *If, moreover, the inclusion* $\{\mathrm{char}({}^\dagger k), \mathrm{char}({}^\ddagger k)\} \subseteq {}^\dagger\Sigma \cup \{0\} = {}^\ddagger\Sigma \cup \{0\}$ [*cf.* (i)] *holds, then the equality*

$$\mathrm{char}({}^\dagger k) = \mathrm{char}({}^\ddagger k)$$

holds.

(v) *If, moreover, the set* ${}^\dagger\Sigma = {}^\ddagger\Sigma$ [*cf.* (i)] *is the set of all prime numbers, and* $\mathrm{char}({}^\dagger k) = \mathrm{char}({}^\ddagger k) \neq 0$ [*cf.* (iv)], *then the equalities*

$${}^\dagger g = {}^\ddagger g, \qquad {}^\dagger r = {}^\ddagger r$$

hold.

Proof. The first equality of assertion (i) follows from the equivalence of (3) $\Leftrightarrow$ (4) of Lemma 4.6. Next, we verify the second equality of assertion (i) and assertion (iii). Let us first observe that since [we have assumed that], for each $\Box \in \{\dagger, \ddagger\}$, the set ${}^\Box\Sigma$ is nonempty, and the inequality $3 \leqslant \#{}^\Box\Sigma$ holds whenever $\mathrm{char}({}^\Box k) \in {}^\Box\Sigma$, it follows from the first equality of assertion (i) that $({}^\dagger\Sigma \setminus ({}^\dagger\Sigma \cap \{\mathrm{char}({}^\dagger k)\})) \cap ({}^\ddagger\Sigma \setminus ({}^\ddagger\Sigma \cap \{\mathrm{char}({}^\ddagger k)\}))$ is nonempty. Let l be an element of this intersection. Then it is immediate that, to verify the second equality of assertion (i) and assertion (iii), we may assume without loss of generality, by replacing ${}^\dagger\Sigma = {}^\ddagger\Sigma$ [cf. the first equality of assertion (i)] by the subset $\{l\}$, that ${}^\dagger\Sigma = {}^\ddagger\Sigma = \{l\}$. Then the second equality of assertion (i) and assertion (iii) follow from [8, Theorem 2.5, (i), (vi)] [cf. also [26, Theorem 2.15]]. This completes the proof of assertions (i), (iii). Assertion (ii) is a formal consequence of Theorem 4.8.

Finally, we verify assertions (iv), (v). Let us first observe that it follows from assertion (ii) that, to verify assertions (iv), (v), we may assume without loss of generality, by replacing ${}^\dagger n = {}^\ddagger n$ [cf. assertion (i)] by 1, that ${}^\dagger n = {}^\ddagger n = 1$. Then since [we have assumed that], for each $\Box \in \{\dagger, \ddagger\}$, the set ${}^\Box\Sigma$ is nonempty, and the inequality $3 \leqslant \#{}^\Box\Sigma$ holds whenever $\mathrm{char}({}^\Box k) \in {}^\Box\Sigma$, assertion (iv) follows from [30, Corollary 1.2]. Moreover, assertion (v) follows from [31, Theorem 4.1]. This completes the proof of assertions (iv), (v), hence also of Corollary 4.9. $\Box$

COROLLARY 4.10. *Suppose that we are in the situation of Corollary* 4.9. *Suppose, moreover, that the following four conditions are satisfied:*

(1) *The equality* ${}^\dagger g = 0$ *holds.*

(2) *Both* ${}^\dagger k$ *and* ${}^\ddagger k$ *are algebraic over the minimal subfields, respectively.*

(3) *The inequality* $\mathrm{char}({}^\dagger k) > 0$ *holds.*

(4) *The set* ${}^\dagger\Sigma$ *is the set of all prime numbers.*

Then there exists a commutative diagram of schemes

$$\begin{array}{ccc} U^{\dagger X}_{(\dagger n)} & \xrightarrow{\sim} & U^{\ddagger X}_{(\ddagger n)} \\ \downarrow & & \downarrow \\ {}^{\dagger}X_{(\dagger n)} & \xrightarrow{\sim} & {}^{\ddagger}X_{(\ddagger n)} \end{array}$$

— *where the horizontal arrows are isomorphisms of schemes, and the vertical arrows are the natural open immersions.*

Proof. This assertion follows immediately from Corollary 4.9, (i), (ii), (iv), and [31], Corollary 5.9. □

5. Cyclotomes that arise from configuration spaces of hyperbolic curves

In the present §5, we discuss cyclotomes that arise from the tame fundamental groups of configuration spaces of hyperbolic curves [cf. Definition 5.4, (iv), below]. In particular, we establish a certain synchronization phenomenon concerning such cyclotomes [cf. Lemma 5.5, (vi), below].

Definition 5.1. We shall define a *finitely generated field* to be a field that is finitely generated over the minimal subfield of the field.

Proposition 5.2. *Let k be a field, $\overline{k}$ a separable closure of k, $X^+ = (X, D)$ a good pair over k, and Π an intermediate profinite quotient of the profinite quotient $\pi_1^{\mathrm{tame}}(X^+) \twoheadrightarrow \mathrm{Gal}(\overline{k}/k)$ [cf. Definition* 1.1, (iii)] *of $\pi_1^{\mathrm{tame}}(X^+)$. Write $\Delta \subseteq \Pi$ for the* [*necessarily closed*] *subgroup of Π obtained by forming the kernel of the resulting continuous surjective homomorphism $\Pi \twoheadrightarrow \mathrm{Gal}(\overline{k}/k)$. Suppose that k is either a finitely generated field or the perfection of a finitely generated field, and that X is proper over k. Then the following assertions hold:*

- (i) *Suppose that k is infinite. Then the topological group $\mathrm{Gal}(\overline{k}/k)$ is quasi-elastic but not topologically finitely generated.*
- (ii) *The topological group Δ is topologically finitely generated.*
- (iii) *The field k is finite if and only if the topological group Π is topologically finitely generated.*
- (iv) *Suppose that k is finite. Then the subgroup $\Delta \subseteq \Pi$ coincides with the kernel of the natural continuous surjective homomorphism from Π to the maximal abelian torsion-free profinite quotient of Π.*
- (v) *Suppose that k is infinite. Then the subgroup $\Delta \subseteq \Pi$ coincides with the unique maximal normal closed subgroup of Π that is topologically finitely generated.*

Proof. First, we verify assertion (i). Let us first recall that it is well-known that the absolute Galois group of a field is isomorphic to the absolute Galois group of the perfection of the field. Thus, assertion (i) follows from [3, Theorem 13.4.2], [3, Lemma 16.11.5], and [3, Proposition 16.11.6]. This completes the proof of assertion (i). Assertion (ii) follows from [20, Proposition 2.2].

Next, we verify assertion (iii). Observe that one verifies easily that it follows from assertion (ii) that, to verify assertion (iii), we may assume without loss of generality, by replacing Π by $\mathrm{Gal}(\overline{k}/k)$, that $\Pi = \mathrm{Gal}(\overline{k}/k)$. If k is finite, then it is well-known that Π is procyclic, hence also topologically finitely generated. If k is infinite, then it follows from assertion (i) that Π is not topologically finitely generated. This completes the proof of assertion (iii). Assertion (iv) follows from [20, Theorem 2.6, (i)]. Assertion (v) follows immediately from assertions (i), (ii). This completes the proof of Proposition 5.2. $\square$

In the remainder of the present §5, for each $\square \in \{\dagger, \ddagger\}$, let

- ${}^{\square}n$ be a positive integer,
- ${}^{\square}g$, ${}^{\square}r$ nonnegative integers such that $2 - 2{}^{\square}g - {}^{\square}r < 0$,
- ${}^{\square}k$ a field,
- ${}^{\square}\overline{k}$ a separable closure of ${}^{\square}k$, and
- ${}^{\square}X^{+} = ({}^{\square}X, {}^{\square}D)$ a hyperbolic curve of type $({}^{\square}g, {}^{\square}r)$ over ${}^{\square}k$.

For each $\square \in \{\dagger, \ddagger\}$ and each $i \in \{0, \ldots, {}^{\square}n\}$, write

- ${}^{\square}p$ for the characteristic of the field ${}^{\square}k$,
- ${}^{\square}X^{+}_{{}^{\square}\overline{k}} \overset{\text{def}}{=} ({}^{\square}X \times_{{}^{\square}k} {}^{\square}\overline{k}, {}^{\square}D \times_{{}^{\square}k} {}^{\square}\overline{k})$ for the hyperbolic curve over ${}^{\square}\overline{k}$ obtained by forming the base-change of ${}^{\square}X^{+}$ to ${}^{\square}\overline{k}$,
- $G_{{}^{\square}k} \overset{\text{def}}{=} \mathrm{Gal}({}^{\square}\overline{k}/{}^{\square}k)$ for the absolute Galois group of the field ${}^{\square}k$ determined by the separable closure ${}^{\square}\overline{k}$,
- ${}^{\square}\Pi_i \overset{\text{def}}{=} \pi_1^{\text{tame}}({}^{\square}X^{+}_{(i)})$ [cf. Definition 2.6; Lemma 2.8, (i)], and
- ${}^{\square}\Delta_i \overset{\text{def}}{=} \pi_1^{\text{tame}}\big(({}^{\square}X^{+}_{{}^{\square}\overline{k}})_{(i)}\big)$ [cf. Definition 2.6; Lemma 2.8, (i)].

Thus, for each $\square \in \{\dagger, \ddagger\}$ and each $i \in \{0, \ldots, {}^{\square}n\}$, the natural morphisms $({}^{\square}X_{{}^{\square}\overline{k}})_{(i)} \to {}^{\square}X_{(i)} \to \mathrm{Spec}({}^{\square}k)$ [cf. Definition 2.6] determine an exact sequence of topological groups

$$1 \longrightarrow {}^{\square}\Delta_i \longrightarrow {}^{\square}\Pi_i \longrightarrow G_{{}^{\square}k} \longrightarrow 1.$$

For each $\square \in \{\dagger, \ddagger\}$, let us fix a lifting $\mathrm{Spec}({}^{\square}k) \to \mathscr{M}_{{}^{\square}g, {}^{\square}r}$ of the classifying morphism $\mathrm{Spec}({}^{\square}k) \to \mathscr{M}_{{}^{\square}g, [{}^{\square}r]}$ of the hyperbolic curve ${}^{\square}X^{+}$ [cf. Remark 2.3.1] whenever the hyperbolic curve ${}^{\square}X^{+}$ is split.

Definition 5.3. Let $\square$ be an element of $\{\dagger, \ddagger\}$. Then, for each positive integer j, we shall write

$$\mu_j({}^{\square}\overline{k}) \subseteq {}^{\square}\overline{k}^{\times}$$

for the subgroup of j-th roots of unity in ${}^{\square}\overline{k}$. Moreover, we shall write

$$\Lambda({}^{\square}\overline{k}) \overset{\text{def}}{=} \varprojlim_{i} \mu_i({}^{\square}\overline{k})$$

— where i ranges over the positive integers — for the *cyclotome* associated to ${}^{\square}\overline{k}$.

Remark 5.3.1. Let $\square$ be an element of $\{\dagger, \ddagger\}$. Then recall that it is well-known that the module $\Lambda({}^{\square}\overline{k})$ has a natural structure of profinite module; moreover, the resulting profinite module is isomorphic to the profinite (respectively, pro-prime-to-${}^{\square}p$) completion of the additive module $\mathbb{Z}$ whenever ${}^{\square}p = 0$ (respectively, ${}^{\square}p \neq 0$).

Definition 5.4. Let $\square$ be an element of $\{\dagger, \ddagger\}$, I a subset of $\{1, \ldots, {}^{\square}n + \varepsilon_{{}^{\square}X^{+}}\}$ [cf. Definition 2.7, (i)] of cardinality $\geqslant \varepsilon_{{}^{\square}X^{+}} + 1$, and i an element of I.

(i) We shall write

$$
{}^{\square}F_{i\in I} \overset{\mathrm{def}}{=} F_{\{1,\dots,{}^{\square}n+\varepsilon_{{}^{\square}X^{+}}\}\setminus(I\setminus\{i\})}({}^{\square}\Delta_{{}^{\square}n})/F_{\{1,\dots,{}^{\square}n+\varepsilon_{{}^{\square}X^{+}}\}\setminus I}({}^{\square}\Delta_{{}^{\square}n})
$$

$$
\subseteq {}^{\square}\Delta_{{}^{\square}n}/F_{\{1,\dots,{}^{\square}n+\varepsilon_{{}^{\square}X^{+}}\}\setminus I}({}^{\square}\Delta_{{}^{\square}n}) = {}^{\square}\Delta_{\#I-\varepsilon_{{}^{\square}X^{+}}}
$$

[cf. Definition 3.4] and

$$
{}^{\square}\underline{F}_{i\in I}
$$

for the identity quotient of ${}^{\square}F_{i\in I}$, i.e., ${}^{\square}\underline{F}_{i\in I} \overset{\mathrm{def}}{=} {}^{\square}F_{i\in I}$ (respectively, for the maximal pro-prime-to-${}^{\square}p$ quotient of ${}^{\square}F_{i\in I}$), whenever ${}^{\square}p = 0$ (respectively, ${}^{\square}p \neq 0$). Observe that one verifies easily from the various definitions involved that this subgroup ${}^{\square}F_{i\in I} \subseteq {}^{\square}\Delta_{\#I-\varepsilon_{{}^{\square}X^{+}}}$ of ${}^{\square}\Delta_{\#I-\varepsilon_{{}^{\square}X^{+}}}$ is a generalized fiber subgroup of ${}^{\square}\Delta_{\#I-\varepsilon_{{}^{\square}X^{+}}}$ of co-length $\#I - \varepsilon_{{}^{\square}X^{+}} - 1$.

(ii) Recall that one verifies immediately from Lemma 1.7 and Lemma 2.8, (ii), that the subgroup ${}^{\square}F_{i\in I} \subseteq {}^{\square}\Delta_{\#I-\varepsilon_{{}^{\square}X^{+}}}$ of ${}^{\square}\Delta_{\#I-\varepsilon_{{}^{\square}X^{+}}}$ may be naturally regarded as a quotient of the tame fundamental group of the good pair obtained by considering the geometric fiber [cf. Lemma 1.4] of the unique morphism ${}^{\square}X_{(\#I-\varepsilon_{{}^{\square}X^{+}})} \to {}^{\square}X_{(\#I-\varepsilon_{{}^{\square}X^{+}}-1)}$ that fits into the commutative diagram

$$
\begin{array}{ccc}
 & {}^{\square}X_{({}^{\square}n)} & \\
 \swarrow^{\mathrm{pr}_{\{1,\dots,{}^{\square}n+\varepsilon_{{}^{\square}X^{+}}\}\setminus I}} & & \searrow^{\mathrm{pr}_{\{1,\dots,{}^{\square}n+\varepsilon_{{}^{\square}X^{+}}\}\setminus(I\setminus\{i\})}} \\
{}^{\square}X_{(\#I-\varepsilon_{{}^{\square}X^{+}})} & \longrightarrow & {}^{\square}X_{(\#I-\varepsilon_{{}^{\square}X^{+}}-1)}.
\end{array}
$$

We shall define a *cuspidal inertia subgroup* of ${}^{\square}F_{i\in I}$ (respectively, of ${}^{\square}\underline{F}_{i\in I}$) to be a [necessarily closed] subgroup of ${}^{\square}F_{i\in I}$ (respectively, of ${}^{\square}\underline{F}_{i\in I}$) obtained by forming the image of a cuspidal inertia subgroup [i.e., an inertia subgroup associated to a cusp] of the tame fundamental group of the good pair obtained by considering the geometric fiber of the above morphism ${}^{\square}X_{(\#I-\varepsilon_{{}^{\square}X^{+}})} \to {}^{\square}X_{(\#I-\varepsilon_{{}^{\square}X^{+}}-1)}$.

(iii) For each positive integer j prime to ${}^{\square}p$, we shall write

$$
H_c^2({}^{\square}\underline{F}_{i\in I}, \mathbb{Z}/j\mathbb{Z})
$$

for the "second cohomology group with compact supports" of ${}^{\square}\underline{F}_{i\in I}$ [cf. [9, Definition 3.1, (ii), (iv)]], i.e., the module defined as follows:

– We shall define a *cuspidally trivialized central extension of* ${}^{\square}\underline{F}_{i\in I}$ *by* $\mathbb{Z}/j\mathbb{Z}$ to be a central extension

$$
1 \longrightarrow \mathbb{Z}/j\mathbb{Z} \longrightarrow E \longrightarrow {}^{\square}\underline{F}_{i\in I} \longrightarrow 1
$$

of ${}^{\square}\underline{F}_{i\in I}$ by $\mathbb{Z}/j\mathbb{Z}$ equipped with a collection $\{s_C \colon C \to E \times_{{}^{\square}\underline{F}_{i\in I}} C\}_C$ — where C ranges over the cuspidal inertia subgroups of ${}^{\square}\underline{F}_{i\in I}$ — of splittings of the extension obtained by restricting the above extension E to $C \subseteq {}^{\square}\underline{F}_{i\in I}$ such that, for each cuspidal inertia subgroup C of ${}^{\square}\underline{F}_{i\in I}$ and each $\gamma \in {}^{\square}\underline{F}_{i\in I}$, the splitting for the conjugate of C by γ is given by the conjugate of the splitting for C by [a lifting in E of] γ.

– Let $(E, \{s_C\}_C)$, $(E', \{s'_C\}_C)$ be cuspidally trivialized central extensions of ${}^{\square}\underline{F}_{i\in I}$ by $\mathbb{Z}/j\mathbb{Z}$. Then we shall say that $(E, \{s_C\}_C)$ is *equivalent* to $(E', \{s'_C\}_C)$ if there exists a continuous isomorphism of E with E' over ${}^{\square}\underline{F}_{i\in I}$ that restricts to the identity automorphism of $\mathbb{Z}/j\mathbb{Z}$ and, moreover, is compatible [in the evident sense] with the collections $\{s_C\}_C$, $\{s'_C\}_C$ of splittings.

– The set $H^2_c({}^{\square}\underline{F}_{i\in I}, \mathbb{Z}/j\mathbb{Z})$ is defined to be the set of the equivalence classes of cuspidally trivialized central extensions of ${}^{\square}\underline{F}_{i\in I}$ by $\mathbb{Z}/j\mathbb{Z}$.
– Let $(E, \{s_C\}_C)$, $(E', \{s'_C\}_C)$ be cuspidally trivialized central extensions of ${}^{\square}\underline{F}_{i\in I}$ by $\mathbb{Z}/j\mathbb{Z}$. Then it is immediate that the fiber product $E \times_{{}^{\square}\underline{F}_{i\in I}} E'$ has a natural structure of a central extension of ${}^{\square}\underline{F}_{i\in I}$ by $\mathbb{Z}/j\mathbb{Z}\times\mathbb{Z}/j\mathbb{Z}$. Thus, by pushing out this central extension by the addition $\mathbb{Z}/j\mathbb{Z}\times\mathbb{Z}/j\mathbb{Z} \to \mathbb{Z}/j\mathbb{Z}$ of the module $\mathbb{Z}/j\mathbb{Z}$, we obtain a central extension E'' of ${}^{\square}\underline{F}_{i\in I}$ by $\mathbb{Z}/j\mathbb{Z}$. Moreover, one verifies immediately that the collections $\{s_C\}_C$, $\{s'_C\}_C$ of splittings naturally determine a collection $\{s''_C\}_C$ of splittings of the extension obtained by restricting the above extension E'' to the various cuspidal inertia subgroups of ${}^{\square}\underline{F}_{i\in I}$, which gives rise to a structure of cuspidally trivialized central extension of ${}^{\square}\underline{F}_{i\in I}$ by $\mathbb{Z}/j\mathbb{Z}$. Now one also verifies immediately that the equivalence class $[E'', \{s''_C\}_C]$ of $(E'', \{s''_C\}_C)$ depends only on the respective equivalence classes $[E, \{s_C\}_C]$, $[E', \{s'_C\}_C]$ of $(E, \{s_C\}_C)$, $(E', \{s'_C\}_C)$. Finally, one also verifies immediately that if one writes

$$[E, \{s_C\}_C] + [E', \{s'_C\}_C] \stackrel{\text{def}}{=} [E'', \{s''_C\}_C],$$

then this "+" determines a module structure on the set $H^2_c({}^{\square}\underline{F}_{i\in I}, \mathbb{Z}/j\mathbb{Z})$.

(iv) We shall write

$$\Lambda({}^{\square}\underline{F}_{i\in I}) \stackrel{\text{def}}{=} \varprojlim_j \mathrm{Hom}\big(H^2_c({}^{\square}\underline{F}_{i\in I}, \mathbb{Z}/j\mathbb{Z}), \mathbb{Z}/j\mathbb{Z}\big)$$

— where j ranges over the positive integers prime to ${}^{\square}p$ — for the *cyclotome* associated to ${}^{\square}\underline{F}_{i\in I}$ [cf. [9, Definition 3.8, (i)]].

Remark 5.4.1. Suppose that we are in the situation of Definition 5.4.

(i) Recall that one verifies immediately from Lemma 2.8, (iii), that if ${}^{\square}p \neq 0$, then the group ${}^{\square}\underline{F}_{i\in I}$ may be naturally identified with the maximal pro-prime-to-${}^{\square}p$ quotient of the étale fundamental group of the geometric fiber of the morphism $U^{{}^{\square}X}_{(\#I-\varepsilon_{{}^{\square}X^+})} \to U^{{}^{\square}X}_{(\#I-\varepsilon_{{}^{\square}X^+}-1)}$ [cf. Definition 2.6] determined by the morphism ${}^{\square}X_{(\#I-\varepsilon_{{}^{\square}X^+})} \to {}^{\square}X_{(\#I-\varepsilon_{{}^{\square}X^+}-1)}$ that appears in Definition 5.4, (ii). In particular, it is well-known [cf., e.g., the discussion preceding [30, Corollary 1.4]] that if C is a cuspidal inertia subgroup of ${}^{\square}F_{i\in I}$ (respectively, of ${}^{\square}\underline{F}_{i\in I}$), then there exists a natural identification isomorphism

$$C \xrightarrow{\sim} \Lambda({}^{\square}\overline{k}).$$

(ii) It follows immediately from [9, Corollary 3.9, (ii), (iii)] that there exists a natural identification isomorphism

$$\Lambda({}^{\square}\underline{F}_{i\in I}) \xrightarrow{\sim} \Lambda({}^{\square}\overline{k}).$$

Lemma 5.5. *Let*

$$\alpha\colon {}^{\dagger}\Pi_{{}^{\dagger}n} \xrightarrow{\sim} {}^{\ddagger}\Pi_{{}^{\ddagger}n}$$

be a continuous isomorphism. Suppose that the following two conditions are satisfied:

(1) *The isomorphism α restricts to an isomorphism ${}^{\dagger}\Delta_{{}^{\dagger}n} \xrightarrow{\sim} {}^{\ddagger}\Delta_{{}^{\ddagger}n}$.*

(2) *There exists a prime number l such that $l \neq {}^{\dagger}p$, and, moreover, the image of the l-adic cyclotomic character $G_{{}^{\dagger}k} \to \mathbb{Z}_l^{\times}$ of $G_{{}^{\dagger}k}$ is open.*

Then the following assertions hold:

(i) *The equalities* ${}^\dagger n = {}^\ddagger n$, ${}^\dagger p = {}^\ddagger p$ *hold.*

(ii) *Let* i *be an element of* $\{0, \dots, {}^\dagger n = {}^\ddagger n\}$ [*cf.* (i)]. *Then the isomorphism* α *determines a bijective map*

$$\mathrm{GFS}_i({}^\dagger\Delta_{{}^\dagger n}) \xrightarrow{\sim} \mathrm{GFS}_i({}^\ddagger\Delta_{{}^\ddagger n})$$

[*cf. Definition* 3.4].

(iii) *Let* ${}^\dagger I$ *be a subset of* $\{1, \dots, {}^\dagger n + \varepsilon_{{}^\dagger X^+_{{}^\dagger \overline{k}}}\}$ *of cardinality* $\geqslant \varepsilon_{{}^\dagger X^+_{{}^\dagger \overline{k}}} + 1$, *and let* ${}^\dagger i$ *be an element of* ${}^\dagger I$. *Then there exist a subset* ${}^\ddagger I$ *of* $\{1, \dots, {}^\ddagger n + \varepsilon_{{}^\ddagger X^+_{{}^\ddagger \overline{k}}}\}$ *of cardinality* $\geqslant \varepsilon_{{}^\ddagger X^+_{{}^\ddagger \overline{k}}} + 1$ *and an element* ${}^\ddagger i$ *of* ${}^\ddagger I$ *such that the isomorphism* α *restricts to continuous isomorphisms*

$$F_{\{1,\dots,{}^\dagger n+\varepsilon_{{}^\dagger X^+_{{}^\dagger \overline{k}}}\}\setminus({}^\dagger I\setminus\{{}^\dagger i\})}({}^\dagger\Delta_{{}^\dagger n}) \xrightarrow{\sim} F_{\{1,\dots,{}^\ddagger n+\varepsilon_{{}^\ddagger X^+_{{}^\ddagger \overline{k}}}\}\setminus({}^\ddagger I\setminus\{{}^\ddagger i\})}({}^\ddagger\Delta_{{}^\ddagger n}),$$

$$F_{\{1,\dots,{}^\dagger n+\varepsilon_{{}^\dagger X^+_{{}^\dagger \overline{k}}}\}\setminus {}^\dagger I}({}^\dagger\Delta_{{}^\dagger n}) \xrightarrow{\sim} F_{\{1,\dots,{}^\ddagger n+\varepsilon_{{}^\ddagger X^+_{{}^\ddagger \overline{k}}}\}\setminus {}^\ddagger I}({}^\ddagger\Delta_{{}^\ddagger n}).$$

In particular, the isomorphism α *determines a continuous isomorphism*

$${}^\dagger F_{{}^\dagger i\in{}^\dagger I} \xrightarrow{\sim} {}^\ddagger F_{{}^\ddagger i\in{}^\ddagger I}.$$

(iv) *In the situation of* (iii), *let* ${}^\dagger C \subseteq {}^\dagger \underline{F}_{{}^\dagger i\in{}^\dagger I}$ *be a cuspidal inertia subgroup of* ${}^\dagger \underline{F}_{{}^\dagger i\in{}^\dagger I}$. *Then the image* ${}^\ddagger C \subseteq {}^\ddagger \underline{F}_{{}^\ddagger i\in{}^\ddagger I}$ *of* ${}^\dagger C \subseteq {}^\dagger \underline{F}_{{}^\dagger i\in{}^\dagger I}$ *by the isomorphism* ${}^\dagger \underline{F}_{{}^\dagger i\in{}^\dagger I} \xrightarrow{\sim} {}^\ddagger \underline{F}_{{}^\ddagger i\in{}^\ddagger I}$ *induced by the isomorphism* ${}^\dagger F_{{}^\dagger i\in{}^\dagger I} \xrightarrow{\sim} {}^\ddagger F_{{}^\ddagger i\in{}^\ddagger I}$ *of* (iii) [*cf.* (i)] *is a cuspidal inertia subgroup of* ${}^\ddagger \underline{F}_{{}^\ddagger i\in{}^\ddagger I}$.

(v) *In the situation of* (iv), *the diagram of modules*

$$\begin{array}{ccccc} {}^\dagger C & \xrightarrow{\sim} & \Lambda({}^\dagger\overline{k}) & \xleftarrow{\sim} & \Lambda({}^\dagger \underline{F}_{{}^\dagger i\in{}^\dagger I}) \\ \wr\downarrow & & & & \downarrow\wr \\ {}^\ddagger C & \xrightarrow{\sim} & \Lambda({}^\ddagger\overline{k}) & \xleftarrow{\sim} & \Lambda({}^\ddagger \underline{F}_{{}^\ddagger i\in{}^\ddagger I}) \end{array}$$

— *where the left-hand horizontal arrows are the isomorphisms discussed in Remark* 5.4.1, (i), *the right-hand horizontal arrows are the isomorphisms discussed in Remark* 5.4.1, (ii), *and the vertical arrows are the isomorphisms induced by the isomorphism* α [*cf.* (i), (iii), (iv)] — *commutes.*

(vi) *In the situation of* (iii), *let* ${}^\dagger I'$ *be a subset of* $\{1, \dots, {}^\dagger n + \varepsilon_{{}^\dagger X^+_{{}^\dagger \overline{k}}}\}$ *of cardinality* $\geqslant \varepsilon_{{}^\dagger X^+_{{}^\dagger \overline{k}}} + 1$, *and let* ${}^\dagger i'$ *be an element of* ${}^\dagger I'$. *Thus, it follows from* (iii) *that there exist a subset* ${}^\ddagger I'$ *of* $\{1, \dots, {}^\ddagger n + \varepsilon_{{}^\ddagger X^+_{{}^\ddagger \overline{k}}}\}$ *of cardinality* $\geqslant \varepsilon_{{}^\ddagger X^+_{{}^\ddagger \overline{k}}} + 1$ *and an element* ${}^\ddagger i'$ *of* ${}^\ddagger I'$ *such that the isomorphism* α *determines a continuous isomorphism* ${}^\dagger F_{{}^\dagger i'\in{}^\dagger I'} \xrightarrow{\sim} {}^\ddagger F_{{}^\ddagger i'\in{}^\ddagger I'}$. *Then the diagram of modules*

$$\begin{array}{ccccc} \Lambda({}^\dagger \underline{F}_{{}^\dagger i\in{}^\dagger I}) & \xrightarrow{\sim} & \Lambda({}^\dagger\overline{k}) & \xleftarrow{\sim} & \Lambda({}^\dagger \underline{F}_{{}^\dagger i'\in{}^\dagger I'}) \\ \wr\downarrow & & & & \downarrow\wr \\ \Lambda({}^\ddagger \underline{F}_{{}^\ddagger i\in{}^\ddagger I}) & \xrightarrow{\sim} & \Lambda({}^\ddagger\overline{k}) & \xleftarrow{\sim} & \Lambda({}^\ddagger \underline{F}_{{}^\ddagger i'\in{}^\ddagger I'}) \end{array}$$

— *where the horizontal arrows are the isomorphisms discussed in Remark* 5.4.1, (ii), *and the vertical arrows are the isomorphisms induced by the isomorphism* α [*cf.* (i), (iii), (iv)] — *commutes.*

Proof. Assertions (i), (ii) follow formally from Corollary 4.9, (i), (ii), (iv) [cf. also condition (1)]. Assertion (iii) follows formally from assertion (ii). Assertion (iv) follows formally from assertion (i) and [17, Corollary 2.7, (i)] [cf. also condition (2)]. Assertion (v) follows formally from [9, Corollary 3.9, (v)].

Finally, we verify assertion (vi). Let us first observe that one verifies easily that, to verify assertion (vi), we may assume without loss of generality, by replacing ${}^{\Box}k$ by a suitable finite extension field of ${}^{\Box}k$ in ${}^{\Box}\overline{k}$, that the hyperbolic curve ${}^{\Box}X^{+}$ is split for each $\Box \in \{\dagger, \ddagger\}$. Moreover, observe that it follows from assertion (ii) [cf. also [9, Corollary 3.9, (ii)]] that, to verify assertion (vi), we may assume without loss of generality, by replacing ${}^{\Box}\Pi_{{}^{\Box}n}$ by the quotient ${}^{\Box}\Pi_{{}^{\Box}n}/F_{{}^{\Box}J}({}^{\Box}\Delta_{{}^{\Box}n}) = {}^{\Box}\Pi_{\min\{{}^{\Box}n, \#\{{}^{\Box}i, {}^{\Box}i'\}\}}$ — where ${}^{\Box}J$ is a suitable subset of $\{1, \ldots, {}^{\Box}n + \varepsilon_{{}^{\Box}X^{+}}\} \setminus \{{}^{\Box}i, {}^{\Box}i'\}$ of cardinality $\max\{0, {}^{\Box}n - \#\{{}^{\Box}i, {}^{\Box}i'\}\}$ — for each $\Box \in \{\dagger, \ddagger\}$, that the inequalities

$$ {}^{\dagger}n \leqslant \#\{{}^{\dagger}i, {}^{\dagger}i'\} \quad (\leqslant 2), \qquad {}^{\ddagger}n \leqslant \#\{{}^{\ddagger}i, {}^{\ddagger}i'\} \quad (\leqslant 2) $$

hold. Next, observe that if ${}^{\dagger}n = 1$, then assertion (vi) is immediate. Thus, to verify assertion (vi), we may assume without loss of generality that the equalities

$$ {}^{\dagger}n = \#\{{}^{\dagger}i, {}^{\dagger}i'\} = {}^{\ddagger}n = \#\{{}^{\ddagger}i, {}^{\ddagger}i'\} = 2 $$

[cf. assertion (i)], hence also the equalities $\{{}^{\dagger}i, {}^{\dagger}i'\} = \{{}^{\ddagger}i, {}^{\ddagger}i'\} = \{1, 2\}$, hold.

Next, let us observe that it follows immediately from the well-known structure of the étale fundamental group of an algebraic curve over a separably closed field of characteristic zero, as well as the well-known structure of the maximal pro-prime-to-p quotient of the étale fundamental group of an algebraic curve over a separably closed field of characteristic $p > 0$, together with the various definitions involved, that, for a given cuspidal inertia subgroup of ${}^{\Box}\underline{F}_{{}^{\Box}i \in {}^{\Box}I}$, the following two conditions are equivalent:

- The given cuspidal inertia subgroup of ${}^{\Box}\underline{F}_{{}^{\Box}i \in {}^{\Box}I}$ arises from the diagonal divisor of ${}^{\Box}X \times_{{}^{\Box}k} {}^{\Box}X$ $(\supseteq U^{{}^{\Box}X}_{({}^{\Box}n)})$ [cf. Remark 2.6.1, (i)].
- The given cuspidal inertia subgroup of ${}^{\Box}\underline{F}_{{}^{\Box}i \in {}^{\Box}I}$ may be obtained by forming the image in ${}^{\Box}\underline{F}_{{}^{\Box}i \in {}^{\Box}I}$ of a closed subgroup of ${}^{\Box}\Delta_{{}^{\Box}n}$ that is a cuspidal inertia subgroup of ${}^{\Box}F_{{}^{\Box}i \in {}^{\Box}I}$ and is also a cuspidal inertia subgroup of ${}^{\Box}F_{{}^{\Box}i' \in {}^{\Box}I}$.

Thus, it follows immediately from assertion (v) that, to verify assertion (vi), it suffices to verify that, in the situation of assertion (iii), if one takes the "${}^{\dagger}C$" to be a cuspidal inertia subgroup that arises from the diagonal divisor of ${}^{\dagger}X \times_{{}^{\dagger}k} {}^{\dagger}X$ $(\supseteq U^{{}^{\dagger}X}_{({}^{\dagger}n)})$, then the "${}^{\ddagger}C$" is a cuspidal inertia subgroup that arises from the diagonal divisor of ${}^{\ddagger}X \times_{{}^{\ddagger}k} {}^{\ddagger}X$ $(\supseteq U^{{}^{\ddagger}X}_{({}^{\ddagger}n)})$. On the other hand, this assertion follows immediately — in light of the well-known structure of the étale fundamental group of an algebraic curve over a separably closed field of characteristic zero, as well as the well-known structure of the maximal pro-prime-to-p quotient of the étale fundamental group of an algebraic curve over a separably closed field of characteristic $p > 0$, together with the various definitions involved — from the above equivalence and assertion (ii). This completes the proof of assertion (vi), hence also of Lemma 5.5. $\Box$

Definition 5.6. In the situation of Lemma 5.5, suppose that the equality $({}^{\dagger}k, {}^{\dagger}\overline{k}) = ({}^{\ddagger}k, {}^{\ddagger}\overline{k})$ holds. Then we shall say that the isomorphism α is *cyclotomically trivial* if, in the situation of Lemma 5.5, (iii), the composite

$$ \Lambda({}^{\dagger}\overline{k}) \xleftarrow{\sim} \Lambda({}^{\dagger}\underline{F}_{{}^{\dagger}i \in {}^{\dagger}I}) \xrightarrow{\sim} \Lambda({}^{\ddagger}\underline{F}_{{}^{\ddagger}i \in {}^{\ddagger}I}) \xrightarrow{\sim} \Lambda({}^{\ddagger}\overline{k}) = \Lambda({}^{\dagger}\overline{k}) $$

— where the first and third arrows are the isomorphisms discussed in Remark 5.4.1, (ii), and the second arrow is the isomorphism induced by the isomorphism of Lemma 5.5, (iii) [cf. also Lemma 5.5, (i), (iv)] — is the identity automorphism of $\Lambda({}^{\dagger}\overline{k})$. Observe that it follows from Lemma 5.5, (vi), that this composite does not depend on the choice of "$({}^{\dagger}I, {}^{\dagger}i, {}^{\ddagger}I, {}^{\ddagger}i)$".

Remark 5.6.1. Let $\alpha\colon {}^{\dagger}\Pi_{{}^{\dagger}n} \xrightarrow{\sim} {}^{\ddagger}\Pi_{{}^{\ddagger}n}$ be a continuous isomorphism. Suppose that the field ${}^{\dagger}k$ (respectively, ${}^{\ddagger}k$) is either a finitely generated field or the perfection of a finitely generated field. Then let us first observe that it follows from Proposition 5.2, (iii), (iv), (v), that the isomorphism α satisfies condition (1) that appears in the statement of Lemma 5.5. Moreover, observe that one verifies easily that the isomorphism α satisfies condition (2) that appears in the statement of Lemma 5.5.

6. The anabelian Grothendieck conjecture for configuration spaces of hyperbolic curves

In the present §6, by applying the results obtained in the previous sections, together with some recent developments in the study of the anabelian geometry of hyperbolic curves over fields of positive characteristic in [32], we prove the absolute version of the anabelian Grothendieck conjecture for the tame fundamental groups of the configuration spaces of hyperbolic curves over [the perfections of] finitely generated fields of positive characteristic [cf. Theorem 6.7 below]. This result may be regarded as a higher dimensional generalization of [30, Theorem 0.5], [18, Theorem 3.2], and [32, Theorem B]. Finally, we also discuss the relative versions of the anabelian Grothendieck conjecture for the tame fundamental groups of the configuration spaces of hyperbolic curves over [the perfections of] finitely generated fields of positive characteristic [cf. Theorem 6.8 below and Corollary 6.9 below], which generalize [28, Theorem 1], [29, Theorem 5.1.3], [32, Theorem A], and [32, Theorem 2.9]. In the present §6, we shall apply the notational conventions introduced at the discussion preceding Definition 5.3. Suppose, moreover, that, for each $\square \in \{\dagger, \ddagger\}$, the characteristic ${}^{\square}p$ is positive except in the discussions of Remark 6.7.2.

Definition 6.1. Let S_1, S_2 be schemes. Then we shall write

$$\mathrm{Isom}(S_1, S_2)$$

for the set of isomorphisms $S_1 \xrightarrow{\sim} S_2$ of schemes and

$$\mathrm{Aut}(S_1) \stackrel{\mathrm{def}}{=} \mathrm{Isom}(S_1, S_1)$$

for the group of automorphisms of the scheme S_1.

Proposition 6.2. *For each $\square \in \{\dagger, \ddagger\}$, write $\mathrm{Aut}^{\mathscr{M}}(U^{{}^{\square}X}_{({}^{\square}n)}) \subseteq \mathrm{Aut}(U^{{}^{\square}X}_{({}^{\square}n)})$ [cf. Definition 2.6] for the subgroup of the [automorphisms of $U^{{}^{\square}X}_{({}^{\square}n)}$ induced by the] modular symmetry automorphisms of ${}^{\square}X_{({}^{\square}n)}$. Then the following assertions hold:*

(i) *Every isomorphism $U^{{}^{\dagger}X}_{({}^{\dagger}n)} \xrightarrow{\sim} U^{{}^{\ddagger}X}_{({}^{\ddagger}n)}$ extends to a unique isomorphism ${}^{\dagger}X_{({}^{\dagger}n)} \xrightarrow{\sim} {}^{\ddagger}X_{({}^{\ddagger}n)}$ [cf. Definition 2.6].*

(ii) *Every isomorphism $U^{{}^{\dagger}X}_{({}^{\dagger}n)} \xrightarrow{\sim} U^{{}^{\ddagger}X}_{({}^{\ddagger}n)}$ lies on a unique isomorphism ${}^{\ddagger}k \xrightarrow{\sim} {}^{\dagger}k$. In the remainder of the statement of the present Proposition 6.2, fix an isomorphism ${}^{\ddagger}\overline{k} \xrightarrow{\sim} {}^{\dagger}\overline{k}$ over this isomorphism ${}^{\ddagger}k \xrightarrow{\sim} {}^{\dagger}k$.*

(iii) *Suppose that the equality ${}^{\dagger}n = {}^{\ddagger}n$ holds, and that, for each $\square \in \{\dagger, \ddagger\}$, the hyperbolic curve ${}^{\square}X^{+}$ is split. Recall from Remark 2.6.1, (i), that, for each $\square \in \{\dagger, \ddagger\}$, the scheme $U^{{}^{\square}X}_{({}^{\square}n)}$ may* *be naturally identified with the* $^{\square}n$*-th configuration space of* $U^{\square X}_{(1)}$*, which thus implies that every isomorphism* $U^{\dagger X}_{(1)} \xrightarrow{\sim} U^{\ddagger X}_{(1)}$ *naturally determines an isomorphism* $U^{\dagger X}_{(^{\dagger}n)} \xrightarrow{\sim} U^{\ddagger X}_{(^{\ddagger}n)}$*. In particular, we have a natural [necessarily injective] map* $\mathrm{Isom}(U^{\dagger X}_{(1)}, U^{\ddagger X}_{(1)}) \hookrightarrow \mathrm{Isom}(U^{\dagger X}_{(^{\dagger}n)}, U^{\ddagger X}_{(^{\ddagger}n)})$*, by means of which we shall regard* $\mathrm{Isom}(U^{\dagger X}_{(1)}, U^{\ddagger X}_{(1)})$ *as a subset of* $\mathrm{Isom}(U^{\dagger X}_{(^{\dagger}n)}, U^{\ddagger X}_{(^{\ddagger}n)})$*:*

$$\mathrm{Isom}(U^{\dagger X}_{(1)}, U^{\ddagger X}_{(1)}) \subseteq \mathrm{Isom}(U^{\dagger X}_{(^{\dagger}n)}, U^{\ddagger X}_{(^{\ddagger}n)}).$$

Let $f\colon U^{\dagger X}_{(^{\dagger}n)} \xrightarrow{\sim} U^{\ddagger X}_{(^{\ddagger}n)}$ *be an isomorphism. Then there exists an element* σ_f *of* $\mathrm{Aut}^{\mathscr{M}}(U^{\dagger X}_{(^{\dagger}n)})$ *such that, for every subset* I *of* $\{1, \ldots, {}^{\dagger}n + \varepsilon_{^{\dagger}X^{+}}\} \cap \{1, \ldots, {}^{\ddagger}n + \varepsilon_{^{\ddagger}X^{+}}\}$ *[cf. Definition 2.7, (i)] of cardinality* $\leqslant {}^{\dagger}n = {}^{\ddagger}n$*, the continuous outer isomorphism* ${}^{\dagger}\Delta_{^{\dagger}n} \xrightarrow{\sim} {}^{\ddagger}\Delta_{^{\ddagger}n}$ *induced by the isomorphism* $U^{\dagger X}_{(^{\dagger}n)} \times_{^{\dagger}k} {}^{\dagger}\overline{k} \xrightarrow{\sim} U^{\ddagger X}_{(^{\ddagger}n)} \times_{^{\ddagger}k} {}^{\ddagger}\overline{k}$ *determined by the composite* $f \circ \sigma_f$ *[cf. (ii)] maps* $F_I({}^{\dagger}\Delta_{^{\dagger}n})$ *[cf. Definition 3.4] bijectively onto* $F_I({}^{\ddagger}\Delta_{^{\ddagger}n})$*. Moreover, in this situation, the composite* $f \circ \sigma_f \in \mathrm{Isom}(U^{\dagger X}_{(^{\dagger}n)}, U^{\ddagger X}_{(^{\ddagger}n)})$ *is contained in the subset* $\mathrm{Isom}(U^{\dagger X}_{(1)}, U^{\ddagger X}_{(1)})$*.*

Proof. Assertions (i), (ii) follow immediately from a similar argument to the argument applied in the proof of [10, Lemma 2.7, (i)]. Assertion (iii) follows immediately from a similar argument to the argument applied in the proof of [10, Lemma 2.7, (i), (ii), (iii)], where we replace "[MzTa], Corollary 6.3" in the proof of [10, Lemma 2.7, (i), (ii), (iii)] by Corollary 4.9, (ii), of the present paper [cf. also Lemma 4.4, (i), of the present paper]. This completes the proof of Proposition 6.2. □

Lemma 6.3. *Let* α *be a continuous automorphism of* ${}^{\dagger}\Pi_{^{\dagger}n}$*, and let* $F \subseteq {}^{\dagger}\Pi_{^{\dagger}n}$ *be a generalized fiber subgroup of* ${}^{\dagger}\Delta_{^{\dagger}n}$ *of co-length* ${}^{\dagger}n - 1$*. Suppose that the following three conditions are satisfied:*

(1) *The inequality* ${}^{\dagger}n \geqslant 2$ *holds.*
(2) *The automorphism* α *of* ${}^{\dagger}\Pi_{^{\dagger}n}$ *preserves* $F \subseteq {}^{\dagger}\Pi_{^{\dagger}n}$ *and induces the identity automorphism of the quotient* ${}^{\dagger}\Pi_{^{\dagger}n}/F = {}^{\dagger}\Pi_{^{\dagger}n-1}$*.*
(3) *The field* ${}^{\dagger}k$ *is either a finitely generated field or the perfection of a finitely generated field.*

Then the following assertions hold:

(i) *Let* $N \subseteq {}^{\dagger}\Pi_{^{\dagger}n}$ *be a normal open subgroup of* ${}^{\dagger}\Pi_{^{\dagger}n}$*. Write* $Z_{^{\dagger}n-1} \to U^{\dagger X}_{(^{\dagger}n-1)}$ *(respectively,* $Z_{^{\dagger}n} \to U^{\dagger X}_{(^{\dagger}n)}$*;* $Z'_{^{\dagger}n} \to U^{\dagger X}_{(^{\dagger}n)}$*) for the finite étale Galois covering that corresponds to the normal open subgroup* $N/(N \cap F) \subseteq {}^{\dagger}\Pi_{^{\dagger}n}/F = {}^{\dagger}\Pi_{^{\dagger}n-1}$ *(respectively,* $N \subseteq {}^{\dagger}\Pi_{^{\dagger}n}$*;* $\alpha(N) \subseteq {}^{\dagger}\Pi_{^{\dagger}n}$*). Let* $\overline{z} \to Z_{^{\dagger}n-1}$ *be a geometric point of* $Z_{^{\dagger}n-1}$*. Write* $(Z_{^{\dagger}n})_{\overline{z}}$*,* $(Z'_{^{\dagger}n})_{\overline{z}}$ *for the respective geometric fibers at* $\overline{z} \to Z_{^{\dagger}n-1}$ *of the natural morphisms* $Z_{^{\dagger}n} \to Z_{^{\dagger}n-1}$*,* $Z'_{^{\dagger}n} \to Z_{^{\dagger}n-1}$ *[cf. (2)] and* $\underline{\pi_1^{\text{ét}}}((Z_{^{\dagger}n})_{\overline{z}})$*,* $\underline{\pi_1^{\text{ét}}}((Z'_{^{\dagger}n})_{\overline{z}})$ *for the respective identity quotients of* $\pi_1^{\text{ét}}((Z_{^{\dagger}n})_{\overline{z}})$*,* $\pi_1^{\text{ét}}((Z'_{^{\dagger}n})_{\overline{z}})$*, i.e.,* $\underline{\pi_1^{\text{ét}}}((Z_{^{\dagger}n})_{\overline{z}}) \stackrel{\text{def}}{=} \pi_1^{\text{ét}}((Z_{^{\dagger}n})_{\overline{z}})$*,* $\underline{\pi_1^{\text{ét}}}((Z'_{^{\dagger}n})_{\overline{z}}) \stackrel{\text{def}}{=} \pi_1^{\text{ét}}((Z'_{^{\dagger}n})_{\overline{z}})$ *(respectively, for the respective maximal pro-prime-to-*${}^{\dagger}p$ *quotients of* $\pi_1^{\text{ét}}((Z_{^{\dagger}n})_{\overline{z}})$*,* $\pi_1^{\text{ét}}((Z'_{^{\dagger}n})_{\overline{z}})$*) [cf. conditions (1), (3) of Definition 1.3; Lemma 2.8, (ii)], whenever* ${}^{\dagger}p = 0$ *(respectively,* ${}^{\dagger}p \neq 0$*). Thus, it follows from Lemma 2.8, (iii), that if* ${}^{\dagger}p = 0$ *(respectively,* ${}^{\dagger}p \neq 0$*), then* $N \cap F$*,* $\alpha(N) \cap F$ *(respectively, the maximal pro-prime-to-*${}^{\dagger}p$ *quotients of* $N \cap F$*,* $\alpha(N) \cap F$*) may be naturally identified with* $\underline{\pi_1^{\text{ét}}}((Z_{^{\dagger}n})_{\overline{z}})$*,* $\underline{\pi_1^{\text{ét}}}((Z'_{^{\dagger}n})_{\overline{z}})$*, respectively. In particular, the automorphism* α *determines a continuous isomorphism* $\underline{\pi_1^{\text{ét}}}((Z_{^{\dagger}n})_{\overline{z}}) \xrightarrow{\sim} \underline{\pi_1^{\text{ét}}}((Z'_{^{\dagger}n})_{\overline{z}})$ *[cf. (2)]. Then this isomorphism* $\underline{\pi_1^{\text{ét}}}((Z_{^{\dagger}n})_{\overline{z}}) \xrightarrow{\sim} \underline{\pi_1^{\text{ét}}}((Z'_{^{\dagger}n})_{\overline{z}})$ *arises from a unique isomorphism* $(Z_{^{\dagger}n})_{\overline{z}} \xrightarrow{\sim} (Z'_{^{\dagger}n})_{\overline{z}}$ *over* $\overline{z}$*.*

(ii) *In the situation of* (i), *suppose that the equality* $N = {}^{\dagger}\Pi_{{}^{\dagger}n}$ *holds* [*which thus implies that* $(Z_{{}^{\dagger}n})_{\overline{z}} = (Z'_{{}^{\dagger}n})_{\overline{z}}$]. *Then the unique isomorphism* $(Z_{{}^{\dagger}n})_{\overline{z}} \xrightarrow{\sim} (Z'_{{}^{\dagger}n})_{\overline{z}}$ *of* (i) *is the identity automorphism of* $(Z_{{}^{\dagger}n})_{\overline{z}} = (Z'_{{}^{\dagger}n})_{\overline{z}}$.

(iii) *The automorphism* α *is* F*-inner.*

Proof. First, we verify assertion (i). Let us first observe that it follows from Remark 5.6.1, together with condition (3), that the automorphism α of ${}^{\dagger}\Pi_{{}^{\dagger}n}$ satisfies conditions (1), (2) that appear in the statement of Lemma 5.5. Next, observe that it follows from Lemma 5.5, (vi), together with conditions (1), (2), that if ${}^{\dagger}p = 0$ (respectively, ${}^{\dagger}p \neq 0$), and one writes $\underline{F} \stackrel{\text{def}}{=} F$ (respectively, $\underline{F}$ for the maximal pro-prime-to-${}^{\dagger}p$ quotient of F), then the automorphism of $\Lambda(\underline{F})$ [cf. Definition 5.4, (iv)] induced by α [cf. Lemma 5.5, (iv)] is the identity automorphism. Thus, one verifies immediately — by considering a suitable quotient [i.e., as discussed in Lemma 2.8, (iii)] of the inverse image of the decomposition subgroup of $N/(N \cap F)$ associated to a closed point of $Z_{{}^{\dagger}n-1}$ by the natural continuous surjective homomorphism $N \twoheadrightarrow N/(N \cap F)$ (respectively, $\alpha(N) \twoheadrightarrow \alpha(N)/(\alpha(N) \cap F) = N/(N \cap F)$) — from [32, Theorem 2.9], together with conditions (2), (3), that the continuous isomorphism $\underline{\pi_1^{\text{ét}}}((Z_{{}^{\dagger}n})_{\overline{z}}) \xrightarrow{\sim} \underline{\pi_1^{\text{ét}}}((Z'_{{}^{\dagger}n})_{\overline{z}})$ that appears in the statement of assertion (i) arises from a unique isomorphism $(Z_{{}^{\dagger}n})_{\overline{z}} \xrightarrow{\sim} (Z'_{{}^{\dagger}n})_{\overline{z}}$ over $\overline{z}$, as desired. This completes the proof of assertion (i).

Next, we verify assertion (ii). Let us observe that it follows from [11, Lemma 2.14, (i)] that, to verify assertion (ii), it suffices to verify that the automorphism $\underline{\pi_1^{\text{ét}}}((Z_{{}^{\dagger}n})_{\overline{z}})^{\{l\}} \xrightarrow{\sim} \underline{\pi_1^{\text{ét}}}((Z'_{{}^{\dagger}n})_{\overline{z}})^{\{l\}}$ induced by the isomorphism $\underline{\pi_1^{\text{ét}}}((Z_{{}^{\dagger}n})_{\overline{z}}) \xrightarrow{\sim} \underline{\pi_1^{\text{ét}}}((Z'_{{}^{\dagger}n})_{\overline{z}})$ that appears in the statement of assertion (i) is inner. On the other hand, this assertion follows from [19, Proposition 1.2, (iii)], together with conditions (1), (2). This completes the proof of assertion (ii). Finally, we verify assertion (iii). Observe that it follows formally from assertions (i), (ii) that the restriction of α to F is inner. Thus, assertion (iii) follows from Theorem 3.7, (ii), and Lemma 6.4 below, together with condition (2). This completes the proof of assertion (iii), hence also of Lemma 6.3. ☐

Lemma 6.4. *Let*

$$1 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow 1$$

be an exact sequence of groups, and let σ *be an automorphism of* G_2. *Suppose that* G_1 *is center-free, and that* σ *induces the respective identity automorphisms of* G_1, G_3. *Then* σ *is the identity automorphism of* G_2.

Proof. Let γ be an element of G_2. Then since σ induces the identity automorphism of G_1, the equalities $\gamma^{-1} \cdot \delta \cdot \gamma = \sigma(\gamma^{-1} \cdot \delta \cdot \gamma) = \sigma(\gamma)^{-1} \cdot \delta \cdot \sigma(\gamma)$ for each $\delta \in G_1$, hence also the inclusion $\sigma(\gamma)\gamma^{-1} \in Z_{G_2}(G_1)$, hold. On the other hand, since σ induces the identity automorphism of G_3, the inclusion $\sigma(\gamma)\gamma^{-1} \in G_1$ holds. In particular, one concludes that $\sigma(\gamma)\gamma^{-1} \in Z_{G_2}(G_1) \cap G_1 = Z(G_1)$, which thus [cf. our assumption that G_1 is center-free] implies that σ is the identity automorphism of G_2, as desired. This completes the proof of Lemma 6.4. ☐

Definition 6.5. We shall say that a hyperbolic curve X^+ over a field k is *isotrivial* if, for an arbitrary separable closure $\overline{k}$ of k, there exist a hyperbolic curve X_0^+ over the separable closure k_0 in $\overline{k}$ of the minimal subfield of k and an isomorphism $X^+ \times_k \overline{k} \xrightarrow{\sim} X_0^+ \times_{k_0} \overline{k}$ over $\overline{k}$.

Definition 6.6. Let G_1, G_2 be profinite groups. Then we shall write

$$\mathrm{OutIsom}(G_1, G_2)$$

for the set of continuous outer isomorphisms $G_1 \xrightarrow{\sim} G_2$ and

$$\mathrm{Out}(G_1) \stackrel{\mathrm{def}}{=} \mathrm{OutIsom}(G_1, G_1)$$

for the group of continuous outer automorphisms of G_1.

THEOREM 6.7. *Suppose that the following two conditions are satisfied:*

(1) *For each $\square \in \{\dagger, \ddagger\}$, the field ${}^{\square}k$ is the perfection of a finitely generated field.*

(2) *If ${}^{\dagger}k$ is infinite, then the hyperbolic curve ${}^{\dagger}X^{+}$ is nonisotrivial.*

Then the natural map

$$\mathrm{Isom}(U^{\dagger X}_{({}^{\dagger}n)}, U^{\ddagger X}_{({}^{\ddagger}n)}) \xrightarrow{\ \sim\ } \mathrm{OutIsom}({}^{\dagger}\Pi_{{}^{\dagger}n}, {}^{\ddagger}\Pi_{{}^{\ddagger}n})$$

[*cf. Proposition* 6.2, (i)] *is bijective.*

Proof. First, we verify the injectivity of the map under consideration. Let f be an automorphism of the scheme $U^{\dagger X}_{({}^{\dagger}n)}$ that induces the trivial continuous outer automorphism of the topological group ${}^{\dagger}\Pi_{{}^{\dagger}n}$. Then it follows immediately from Proposition 6.2, (iii), that the automorphism f of $U^{\dagger X}_{({}^{\dagger}n)}$ arises from a unique automorphism f_1 of $U^{\dagger X}_{(1)}$. [Observe that, in this situation, since f induces the trivial continuous outer automorphism of the topological group ${}^{\dagger}\Pi_{{}^{\dagger}n}$, one verifies easily that the "σ_f" of Proposition 6.2, (iii), to be the identity automorphism.] Now observe that since the automorphism f induces the trivial continuous outer automorphism of the topological group ${}^{\dagger}\Pi_{{}^{\dagger}n}$, the automorphism f_1 induces the trivial continuous outer automorphism of the topological group ${}^{\dagger}\Pi_1$. Thus, it follows from [30, Theorem 0.5], [18, Theorem 3.2], and [32, Theorem B] [cf. also conditions (1), (2)] that the automorphism f_1, hence also the automorphism f, is trivial, as desired. This completes the proof of the injectivity of the map under consideration.

Next, we verify the surjectivity of the map under consideration. Let $\alpha\colon {}^{\dagger}\Pi_{{}^{\dagger}n} \xrightarrow{\sim} {}^{\ddagger}\Pi_{{}^{\ddagger}n}$ be a continuous isomorphism. Let us first observe that it follows from Remark 5.6.1, together with condition (1), that the isomorphism α satisfies conditions (1), (2) that appear in the statement of Lemma 5.5. Next, observe that it follows from Lemma 5.5, (i), that the equality ${}^{\dagger}n = {}^{\ddagger}n$ holds. Write $n \stackrel{\mathrm{def}}{=} {}^{\dagger}n = {}^{\ddagger}n$. In the remainder of the present proof, we prove the existence of an isomorphism $U^{\dagger X}_{({}^{\dagger}n)} \xrightarrow{\sim} U^{\ddagger X}_{({}^{\ddagger}n)}$ whose image by the map under consideration is given by the continuous outer automorphism determined by α by induction on n. If $n = 1$, then the desired existence follows from [30, Theorem 0.5], [18, Theorem 3.2], and [32, Theorem B] [cf. also conditions (1), (2)]. Suppose that $n \geqslant 2$, and that the induction hypothesis is in force. Also, observe that one verifies immediately from the injectivity of the map under consideration [i.e., already verified in the first paragraph of the present proof] that, to verify the desired existence, we may assume without loss of generality, by replacing ${}^{\square}k$ by a suitable finite extension field of ${}^{\square}k$ in ${}^{\square}\overline{k}$, that the hyperbolic curve ${}^{\square}X^{+}$ is split for each $\square \in \{\dagger, \ddagger\}$.

Next, observe that it follows immediately from Lemma 5.5, (ii), that the set $\mathrm{OutIsom}({}^{\dagger}\Pi_1, {}^{\ddagger}\Pi_1)$, hence [cf. the induction hypothesis] also the set $\mathrm{Isom}(U^{\dagger X}_{(1)}, U^{\ddagger X}_{(1)})$, is nonempty. Thus, it is immediate that, to verify the desired existence, we may assume without loss of generality, by replacing ${}^{\ddagger}X^{+}$ by ${}^{\dagger}X^{+}$ [cf. also Proposition 6.2, (i), (iii)], that the equality $\dagger = \ddagger$ holds, which thus implies that α is a continuous automorphism of the topological group ${}^{\dagger}\Pi_{{}^{\dagger}n}$. Next, observe that it follows immediately from Lemma 5.5, (ii), that, to verify the desired existence, we may assume without loss of generality, by replacing α by the composite of α with a continuous automorphism of ${}^{\dagger}\Pi_{{}^{\dagger}n}$ that arises from a suitable modular symmetry automorphism of ${}^{\dagger}X_{({}^{\dagger}n)}$, that

(a) the automorphism α preserves every generalized fiber subgroup of ${}^{\dagger}\Delta_{{}^{\dagger}n}$.

Let $F \subseteq {}^{\dagger}\Pi_{{}^{\dagger}n}$ be a generalized fiber subgroup of ${}^{\dagger}\Delta_{{}^{\dagger}n}$ of co-length 1. Thus, it follows immediately from the induction hypothesis [cf. also (a)] that, to verify the desired existence, we may assume without loss of generality, by replacing α by the composite of α with a continuous automorphism of ${}^{\dagger}\Pi_{{}^{\dagger}n}$ that arises from the automorphism of $U^{{}^{\dagger}X}_{({}^{\dagger}n)}$ determined by a suitable automorphism of $U^{{}^{\dagger}X}_{(1)}$ [cf. Proposition 6.2, (i), (iii)], that

(b) the automorphism α induces the identity automorphism of the quotient ${}^{\dagger}\Pi_{{}^{\dagger}n}/F = {}^{\dagger}\Pi_1$.

Next, observe that it is immediate that, to verify the desired existence, it suffices to verify that α is ${}^{\dagger}\Delta_{{}^{\dagger}n}$-inner. In the remainder of the present proof, we prove that α is ${}^{\dagger}\Delta_{{}^{\dagger}n}$-inner by induction on n. If $n = 2$, then it follows from Lemma 6.3, (iii), together with conditions (1), (2) [cf. also (b)], that α is ${}^{\dagger}\Delta_{{}^{\dagger}n}$-inner. Suppose that $n \geqslant 3$, and that the induction hypothesis is in force. Let $F' \subseteq {}^{\dagger}\Pi_{{}^{\dagger}n}$ be a generalized fiber subgroup of ${}^{\dagger}\Delta_{{}^{\dagger}n}$ of co-length ${}^{\dagger}n-1$ that is contained in F. Thus, it follows immediately from the induction hypothesis [cf. also (a)] that

(c) the automorphism of the quotient ${}^{\dagger}\Pi_{{}^{\dagger}n}/F' = {}^{\dagger}\Pi_{{}^{\dagger}n-1}$ induced by α is ${}^{\dagger}\Delta_{{}^{\dagger}n-1}$-inner.

In particular, it follows from Lemma 6.3, (iii), together with conditions (1), (2) [cf. also (c)], that α is ${}^{\dagger}\Delta_{{}^{\dagger}n}$-inner, as desired. This completes the proof of the surjectivity of the map under consideration, hence also of Theorem 6.7. □

Remark 6.7.1. Observe that if, in the situation of Theorem 6.7, one drops the nonisotriviality assumption [i.e., condition (2)], then the conclusion no longer holds in general. A counter-example is discussed in [32, Remark 4.10.1].

Remark 6.7.2. As we discussed so far, the main topic of the present paper is the anabelian Grothendieck conjecture for the configuration spaces associated to hyperbolic curves. In the present Remark, in the context of comparison with that, we would like to discuss the anabelian Grothendieck conjecture for the fiber products of finitely many hyperbolic curves. To this end, suppose that we are in the situation of the discussion preceding Definition 5.3. Suppose, moreover, that, for each $i \in \{1, \ldots, {}^{\square}n\}$ and $\square \in \{\dagger, \ddagger\}$, we are given a hyperbolic curve ${}^{\square}X_i^{+} = ({}^{\square}X_i, {}^{\square}D_i)$ over ${}^{\square}k$. For each $i \in \{1, \ldots, {}^{\square}n\}$ and $\square \in \{\dagger, \ddagger\}$, write ${}^{\dagger}U_i \overset{\text{def}}{=} {}^{\square}X_i \setminus {}^{\square}D_i$. Moreover, for each $\square \in \{\dagger, \ddagger\}$, write

- ${}^{\square}P$ for the fiber product over ${}^{\square}k$ of the ${}^{\square}X_i$'s, where i ranges over the elements of $\{1, \ldots, {}^{\square}n\}$,
- ${}^{\square}E \subseteq {}^{\square}P$ for the reduced closed subscheme of ${}^{\square}P$ whose underlying closed subset is given by the union of the pull-back of ${}^{\square}D_i \subseteq {}^{\square}X_i$ by the projection morphism ${}^{\square}P \to {}^{\square}X_i$, where i ranges over the elements of $\{1, \ldots, {}^{\square}n\}$,
- ${}^{\square}P^{+}$ for the [necessarily good — cf. Lemma 2.8, (i)] pair $({}^{\square}P, {}^{\square}E)$, and
- ${}^{\square}V$ for the fiber product over ${}^{\square}k$ of the ${}^{\square}U_i$'s, where i ranges over the elements of $\{1, \ldots, {}^{\square}n\}$, i.e., the complement ${}^{\square}P \setminus {}^{\square}E$ of ${}^{\square}E$ in ${}^{\square}P$.

Then, by applying a similar result to Proposition 6.2, (i), we obtain a natural map

$$\Phi_{{}^{\dagger}P^{+}, {}^{\ddagger}P^{+}} \colon \mathrm{Isom}({}^{\dagger}V, {}^{\ddagger}V) \longrightarrow \mathrm{OutIsom}\big(\pi_1^{\mathrm{tame}}({}^{\dagger}P^{+}), \pi_1^{\mathrm{tame}}({}^{\ddagger}P^{+})\big)$$

[cf. Definition 1.1, (iii)].

(i) Observe that the above natural map $\Phi_{{}^{\dagger}P^{+}, {}^{\ddagger}P^{+}}$ is not bijective in general even in the case where the fields ${}^{\dagger}k$, ${}^{\ddagger}k$ are finite. An example that violates the bijectivity of the map

$\Phi_{{}^\dagger P^+,{}^\ddagger P^+}$ may be given as follows: First, observe that one verifies easily that there exist a prime number p and a projective smooth curve C over the finite field $\mathbb{F}_{p^2}$ of cardinality p^2 such that if one writes C^F for the base-change of C by the p-th power Frobenius automorphism of $\mathbb{F}_{p^2}$, then C is not isomorphic to C^F over $\mathbb{F}_{p^2}$. [Note that, in this situation, it is immediate that C^F is isomorphic to C as an abstract scheme.] We take

- the positive integers ${}^\dagger n$ and ${}^\ddagger n$ to be 2,
- the fields ${}^\dagger k$ and ${}^\ddagger k$ to be $\mathbb{F}_{p^2}$,
- the fields ${}^\dagger \overline{k}$ and ${}^\ddagger \overline{k}$ to be a fixed algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_{p^2}$,
- the hyperbolic curves ${}^\dagger X_1^+$, ${}^\ddagger X_1^+$, and ${}^\ddagger X_2^+$ to be the hyperbolic curve $(C, \emptyset)$ over ${}^\dagger k$ $(= {}^\ddagger k)$, and
- the hyperbolic curve ${}^\dagger X_2^+$ to be the hyperbolic curve $(C^F, \emptyset)$ over ${}^\dagger k$ $(= {}^\ddagger k)$.

Then it follows from the discussion of the final paragraph of [34, Éxpose I, §11] that

(a) the natural projection morphism $C^F \xrightarrow{\sim} C$ [cf. the definition of C^F] determines a continuous outer isomorphism $\alpha\colon \pi_1^{\mathrm{tame}}({}^\dagger X_2^+) \xrightarrow{\sim} \pi_1^{\mathrm{tame}}({}^\ddagger X_2^+)$.

Now observe that since [it is immediate that] every automorphism of ${}^\dagger \overline{k}$ $(= {}^\ddagger \overline{k})$ commutes with the p-th power Frobenius automorphism of $\overline{\mathbb{F}}_p$,

(b) the outer isomorphism $\alpha\colon \pi_1^{\mathrm{tame}}({}^\dagger X_2^+) \xrightarrow{\sim} \pi_1^{\mathrm{tame}}({}^\ddagger X_2^+)$ of (a) is an outer isomorphism over $G_{{}^\dagger k}$ $(= G_{{}^\ddagger k})$.

Next, recall from Lemma 2.8, (i), [5, Proposition 3], and [5, Proposition B.7] [cf. also conditions (1), (5) of Definition 1.3] that,

(c) for each $\Box \in \{\dagger, \ddagger\}$, the various projection morphisms ${}^\Box P \to {}^\Box X_i$, where i ranges over the elements of $\{1, 2\}$, determine a continuous outer isomorphism of $\pi_1^{\mathrm{tame}}({}^\Box P^+)$ with the fiber product over $G_{{}^\Box k}$ of the $\pi_1^{\mathrm{tame}}({}^\Box X_i^+)$'s, where i ranges over the elements of $\{1, 2\}$.

In particular, it follows from (b), (c) that the identity outer automorphism of $\pi_1^{\mathrm{tame}}({}^\dagger X_1^+) = \pi_1^{\mathrm{tame}}({}^\ddagger X_1^+)$ and the outer isomorphism $\alpha\colon \pi_1^{\mathrm{tame}}({}^\dagger X_2^+) \xrightarrow{\sim} \pi_1^{\mathrm{tame}}({}^\ddagger X_2^+)$ of (a) determine a continuous outer isomorphism $\beta\colon \pi_1^{\mathrm{tame}}({}^\dagger P^+) \xrightarrow{\sim} \pi_1^{\mathrm{tame}}({}^\ddagger P^+)$. In the remainder of the present argument of (i), we verify that

(∗) this outer isomorphism β does not arise from any isomorphism ${}^\dagger V \xrightarrow{\sim} {}^\ddagger V$ [i.e., $C \times_{{}^\dagger k} C \xrightarrow{\sim} C \times_{{}^\dagger k} C^F$] of schemes.

To this end, assume that this outer isomorphism β arises from an isomorphism $f\colon {}^\dagger V \xrightarrow{\sim} {}^\ddagger V$ of schemes. Then observe that it follows from the construction of β that, for each $i \in \{1, 2\}$, the composite $\beta\colon \pi_1^{\mathrm{tame}}({}^\dagger P^+) \xrightarrow{\sim} \pi_1^{\mathrm{tame}}({}^\ddagger X_i^+)$ of $\beta\colon \pi_1^{\mathrm{tame}}({}^\dagger P^+) \xrightarrow{\sim} \pi_1^{\mathrm{tame}}({}^\ddagger P^+)$ and the i-th projection homomorphism $\pi_1^{\mathrm{tame}}({}^\ddagger P^+) \twoheadrightarrow \pi_1^{\mathrm{tame}}({}^\ddagger X_i^+)$ [cf. (c)] factors through the i-th projection homomorphism $\pi_1^{\mathrm{tame}}({}^\dagger P^+) \twoheadrightarrow \pi_1^{\mathrm{tame}}({}^\dagger X_i^+)$ [cf. (c)]. Thus, one verifies immediately that, for each $i \in \{1, 2\}$, the composite ${}^\dagger V \to {}^\ddagger U_i$ of $f\colon {}^\dagger V \xrightarrow{\sim} {}^\ddagger V$ and the i-th projection morphism ${}^\ddagger V \to {}^\ddagger U_i$ factors through the i-th projection morphism ${}^\dagger V \to {}^\dagger U_i$. In particular, one may conclude, by applying a similar argument to this argument to the inverse of f, that there exist isomorphisms $f_1\colon {}^\dagger U_1 \xrightarrow{\sim} {}^\ddagger U_1$, $f_2\colon {}^\dagger U_2 \xrightarrow{\sim} {}^\ddagger U_2$ of schemes such that the isomorphism f is determined by these isomorphisms f_1 and f_2. Now observe that since ${}^\dagger k$ is of cardinality p^2, it follows from [30, Lemma 4.2] that the isomorphism $f_1\colon {}^\dagger U_1 \xrightarrow{\sim} {}^\ddagger U_1$ [i.e., $C \xrightarrow{\sim} C$] is an isomorphism either over ${}^\dagger k$ or over the p-th power Frobenius automorphism of ${}^\dagger k$. Thus, since [we have assumed that] C is not isomorphic to C^F over $\mathbb{F}_{p^2}$, f_1 is an isomorphism over ${}^\dagger k$. On the other hand, since f_1 and f_2 determine an isomorphism f, this implies that $f_2\colon {}^\dagger U_2 \xrightarrow{\sim} {}^\ddagger U_2$ [i.e., $C^F \xrightarrow{\sim} C$] has to be an isomorphism over ${}^\dagger k$, which contradicts our assumption that C is not isomorphic to C^F over $\mathbb{F}_{p^2}$. This completes the proof of assertion $(*)$. In particular, one concludes that the map $\Phi_{{}^\dagger P^+, {}^\ddagger P^+}$ is not bijective in this situation.

(ii) On the other hand, if the fields ${}^\dagger k$, ${}^\ddagger k$ are finitely generated and of characteristic zero, then the above natural map $\Phi_{{}^\dagger P^+, {}^\ddagger P^+}$ is bijective. This bijectivity may be proved as follows: First, to verify the injectivity of the map $\Phi_{{}^\dagger P^+, {}^\ddagger P^+}$, let f be an automorphism of the scheme ${}^\dagger V$ that induces the trivial outer automorphism of the topological group $\pi_1^{\mathrm{tame}}({}^\dagger P^+)$. Then one concludes — by applying, for example, [12, Theorem in Introduction], i.e., in the case where condition (3) is satisfied, to the automorphism of ${}^\dagger k$ induced by f [cf. [30, Lemma 4.2]] — that f is an automorphism over ${}^\dagger k$. Thus, the triviality of f follows immediately from [16, Theorem A]. Next, to verify the surjectivity of the map $\Phi_{{}^\dagger P^+, {}^\ddagger P^+}$, let $\alpha \colon \pi_1^{\mathrm{tame}}({}^\dagger P^+) \xrightarrow{\sim} \pi_1^{\mathrm{tame}}({}^\ddagger P^+)$ be a continuous isomorphism. Then observe that it follows immediately from [20, Corollary 2.8, (i)], as well as a similar argument to the argument applied in the proof of [7, Claim 5.6.A] [cf. also [7, Lemma 5.5]], that the continuous isomorphism α is an isomorphism over some continuous isomorphism $G_{{}^\dagger k} \xrightarrow{\sim} G_{{}^\ddagger k}$. Next, observe that it follows from the main theorem of [23] in the case of characteristic zero [i.e., [6, Proposition 3.19, (ii)]] that this continuous isomorphism $G_{{}^\dagger k} \xrightarrow{\sim} G_{{}^\ddagger k}$ arises from an isomorphism ${}^\ddagger\overline{k} \xrightarrow{\sim} {}^\dagger\overline{k}$ of fields. [Note that [23] has not yet been published. On the other hand, the main theorem of [23] in the case of characteristic zero, i.e., [6, Proposition 3.19, (ii)], may also be derived from [22, Observation in p.146] and [16, Theorem B].] In particular, to verify the fact that α is contained in the image of the map $\Phi_{{}^\dagger P^+, {}^\ddagger P^+}$, we may assume without loss of generality, by replacing ${}^\dagger V$ by the base-change of ${}^\dagger V$ via the isomorphism ${}^\ddagger k \xrightarrow{\sim} {}^\dagger k$ of fields determined by this isomorphism ${}^\ddagger\overline{k} \xrightarrow{\sim} {}^\dagger\overline{k}$ of fields, that the equality $({}^\dagger k, {}^\dagger\overline{k}) = ({}^\ddagger k, {}^\ddagger\overline{k})$ holds, and, moreover, the isomorphism α is an isomorphism over $G_{{}^\dagger k}$. Then it follows immediately from [16, Theorem A] that α is contained in the image of the map $\Phi_{{}^\dagger P^+, {}^\ddagger P^+}$, as desired. This completes the proof of the bijectivity of the map $\Phi_{{}^\dagger P^+, {}^\ddagger P^+}$ in the case where the fields ${}^\dagger k$, ${}^\ddagger k$ are finitely generated and of characteristic zero.

(iii) Let us point out that, as far as the authors know, Theorem 6.7 is the first result concerning the absolute version of the anabelian Grothendieck conjecture for varieties in positive characteristic of higher dimension, i.e., of dimension greater than one [cf. also the argument of (i)].

Theorem 6.8. *Suppose that the following two conditions are satisfied:*

(1) *The equality* $({}^\dagger k, {}^\dagger\overline{k}) = ({}^\ddagger k, {}^\ddagger\overline{k})$ *holds.*

(2) *The field* ${}^\dagger k$ *is either a finitely generated field or the perfection of a finitely generated field.*

Write

$$\mathrm{Isom}_{{}^\dagger k}(U^{{}^\dagger X}_{({}^\dagger n)}, U^{{}^\ddagger X}_{({}^\ddagger n)}) \subseteq \mathrm{Isom}(U^{{}^\dagger X}_{({}^\dagger n)}, U^{{}^\ddagger X}_{({}^\ddagger n)})$$

for the subset of isomorphisms $U^{{}^\dagger X}_{({}^\dagger n)} \xrightarrow{\sim} U^{{}^\ddagger X}_{({}^\ddagger n)}$ *over* ${}^\dagger k = {}^\ddagger k$,

$$\mathrm{Isom}^{\Lambda}_{G_{{}^\dagger k}}({}^\dagger\Pi_{{}^\dagger n}, {}^\ddagger\Pi_{{}^\ddagger n})$$

for the set of continuous isomorphisms ${}^\dagger\Pi_{{}^\dagger n} \xrightarrow{\sim} {}^\ddagger\Pi_{{}^\ddagger n}$ *over* $G_{{}^\dagger k} = G_{{}^\ddagger k}$ *that are cyclotomically trivial* [*cf.* (1), *Remark* 5.6.1], *and*

$${}_{\Delta}\backslash\mathrm{Isom}^{\Lambda}_{G_{{}^\dagger k}}({}^\dagger\Pi_{{}^\dagger n}, {}^\ddagger\Pi_{{}^\ddagger n})$$

for the quotient set of $\mathrm{Isom}^{\Lambda}_{G_{\dagger k}}({}^{\dagger}\Pi_{\dagger n}, {}^{\ddagger}\Pi_{\ddagger n})$ *with respect to* ${}^{\ddagger}\Delta_{\ddagger n}$*-conjugation. Then the natural map*

$$\mathrm{Isom}_{\dagger k}(U^{\dagger X}_{(\dagger n)}, U^{\ddagger X}_{(\ddagger n)}) \xrightarrow{\sim} {}_{\Delta\backslash}\mathrm{Isom}^{\Lambda}_{G_{\dagger k}}({}^{\dagger}\Pi_{\dagger n}, {}^{\ddagger}\Pi_{\ddagger n})$$

[*cf. Proposition* 6.2, (i)] *is bijective.*

Proof. This assertion follows immediately from the argument obtained by replacing "[30, Theorem 0.5], [18, Theorem 3.2], and [32, Theorem B]" in the proof of Theorem 6.7 by [32, Theorem 2.9]. □

Corollary 6.9. *Suppose that the following three conditions are satisfied:*

(1) *The equality* $({}^{\dagger}k, {}^{\dagger}\overline{k}) = ({}^{\ddagger}k, {}^{\ddagger}\overline{k})$ *holds.*

(2) *The field* ${}^{\dagger}k$ *is an infinite finitely generated field.*

(3) *The hyperbolic curve* ${}^{\dagger}X^{+}$ *is nonisotrivial.*

Write

$$\mathrm{Isom}_{\dagger k, F^{-1}_{\dagger k}}(U^{\dagger X}_{(\dagger n)}, U^{\ddagger X}_{(\ddagger n)})$$

for the set of isomorphisms $U^{\dagger X}_{(\dagger n)} \xrightarrow{\sim} U^{\ddagger X}_{(\ddagger n)}$ *in the category* $\mathsf{Var}_{\dagger k, F^{-1}_{\dagger k}}$ *defined in the discussion "Inverting Frobenius" following [29, Lemma 4.1.1],*

$$\mathrm{Isom}_{G_{\dagger k}}({}^{\dagger}\Pi_{\dagger n}, {}^{\ddagger}\Pi_{\ddagger n})$$

for the set of continuous isomorphisms ${}^{\dagger}\Pi_{\dagger n} \xrightarrow{\sim} {}^{\ddagger}\Pi_{\ddagger n}$ *over* $G_{\dagger k} = G_{\ddagger k}$*, and*

$${}_{\Delta\backslash}\mathrm{Isom}_{G_{\dagger k}}({}^{\dagger}\Pi_{\dagger n}, {}^{\ddagger}\Pi_{\ddagger n})$$

for the quotient set of $\mathrm{Isom}_{G_{\dagger k}}({}^{\dagger}\Pi_{\dagger n}, {}^{\ddagger}\Pi_{\ddagger n})$ *with respect to* ${}^{\ddagger}\Delta_{\ddagger n}$*-conjugation. Then the natural map*

$$\mathrm{Isom}_{\dagger k, F^{-1}_{\dagger k}}(U^{\dagger X}_{(\dagger n)}, U^{\ddagger X}_{(\ddagger n)}) \xrightarrow{\sim} {}_{\Delta\backslash}\mathrm{Isom}_{G_{\dagger k}}({}^{\dagger}\Pi_{\dagger n}, {}^{\ddagger}\Pi_{\ddagger n})$$

[*cf. Theorem* 3.7, (ii)*; Proposition* 6.2, (i)*; the discussion following [29, Lemma 4.1.6]*] *is bijective.*

Proof. Let us first observe that one verifies immediately from Theorem 6.8 [cf. also [32, Lemma 4.2]] that, to verify Corollary 6.9, it suffices to verify that, for each continuous isomorphism $\alpha\colon {}^{\dagger}\Pi_{\dagger n} \xrightarrow{\sim} {}^{\ddagger}\Pi_{\ddagger n}$ over $G_{\dagger k} = G_{\ddagger k}$, the composite

$$\Lambda({}^{\dagger}\overline{k}) \xleftarrow{\sim} \Lambda({}^{\dagger}\underline{F}_{\dagger i \in \dagger I}) \xrightarrow{\sim} \Lambda({}^{\ddagger}\underline{F}_{\ddagger i \in \ddagger I}) \xrightarrow{\sim} \Lambda({}^{\ddagger}\overline{k}) = \Lambda({}^{\dagger}\overline{k})$$

discussed in Definition 5.6 that arises from α [cf. conditions (1), (2); Remark 5.6.1] is given by multiplication by ${}^{\dagger}p^{N} = {}^{\ddagger}p^{N}$ for some integer N [cf. also Remark 5.3.1]. On the other hand, this follows immediately from [32, Proposition 4.4], together with Lemma 5.5, (ii) [cf. conditions (1), (2), (3)]. This completes the proof of Corollary 6.9. □

Remark 6.9.1. The "general formal content" of the two remarks following Theorem 6.7 applies to the respective situations discussed in Theorem 6.8, Corollary 6.9, as well. We leave the routine details of translating these remarks into the language of the respective situations of Theorem 6.8, Corollary 6.9 to the interested reader.

Appendix A. Generalized fiber subgroups of configuration space groups

In the present §A, in order to meet the request from a referee, we repeat the purely "group-theoretic algorithm" that constructs the generalized fiber subgroups of configuration space groups discussed in [8, Theorem 2.5] [cf. also [27, Theorem 5.18]].

Definition A.1. Let n be a positive integer, and let l be a prime number.

(i) We shall write $\mathscr{L}_{(0,3,n)}$ for the graded Lie algebra over $\mathbb{Q}_l$ defined by the generators $Z_{i,j}$ — where (i,j) ranges over the pairs of positive integers such that $i < j \leqslant n+3$ — all of which are of weight one, and the relations
  - $\sum_{s=1}^{i-1} Z_{s,i} + \sum_{t=i+1}^{n+3} Z_{i,t} = 0$ for every $i \in \{1,\dots,n+3\}$ and
  - $[Z_{a,b}, Z_{c,d}] = 0$ for every $\{a,b,c,d\} \subseteq \{1\dots n+3\}$ such that $a < b$, $c < d$, and $\{a,b\} \cap \{c,d\} = \emptyset$.

(ii) Let S be a subset of $\{1,\dots,n+3\}$ of cardinality four. Then we shall write $W^S_{(0,3,n)} \subseteq \mathscr{L}_{(0,3,n)}$ for the vector subspace of $\mathscr{L}_{(0,3,n)}$ over $\mathbb{Q}_l$ generated by $Z_{i,j} \in \mathscr{L}_{(0,3,n)}$ — where (i,j) ranges over the pairs of positive integers such that $i < j \leqslant n+3$ and $\{i,j\} \nsubseteq S$.

(iii) We shall write $\mathscr{L}_{(1,1,n)}$ for the graded Lie algebra over $\mathbb{Q}_l$ defined by the generators $X^{(u)}$, $Y^{(u)}$ — where u is an element of $\{1,\dots,n\}$ — all of which are of weight one, and $Z_t^{(u)}$ — where t, u are elements of $\{1,\dots,1+n\}$, $\{1,\dots,n\}$, respectively — all of which are of weight two, and the relations
  - $[X^{(u)}, Y^{(u)}] + \sum_{t=1}^{1+n} Z_t^{(u)} = 0$ for every $u \in \{1,\dots,n\}$,
  - $Z_{1+u}^{(u)} = 0$ for every $u \in \{1,\dots,n\}$,
  - $Z_{1+u'}^{(u)} = Z_{1+u}^{(u')}$ for every $\{u,u'\} \subseteq \{1,\dots,n\}$,
  - $[Z_t^{(u)}, Z_{t'}^{(u')}] = 0$ for every $\{t,t'\} \subseteq \{1,\dots,1+n\}$, $\{u,u'\} \subseteq \{1,\dots,n\}$ such that $\{1+u,t\} \cap \{1+u',t'\} = \emptyset$,
  - $[Z_t^{(u)}, X^{(u')}] = [Z_t^{(u)}, Y^{(u')}] = 0$ for every $t \in \{1,\dots,1+n\}$, $\{u,u'\} \in \{1,\dots,n\}$ such that $u \neq u'$ and $t \neq 1+u'$, and
  - $[X^{(u)}, X^{(u')}] = [X^{(u)}, Y^{(u')}] - Z_{1+u}^{(u')} = [Y^{(u)}, Y^{(u')}] = 0$ for every $\{u,u'\} \in \{1,\dots,n\}$ such that $u \neq u'$.

(iv) Let S be a subset of $\{1,\dots,n+1\}$ of cardinality two. Then we shall write $W^S_{(1,1,n)} \subseteq \mathscr{L}_{(1,1,n)}$ for the vector subspace of $\mathscr{L}_{(1,1,n)}$ over $\mathbb{Q}_l$ generated by $X^{(i)}$ and $Y^{(i)}$ — where i ranges over the elements of $\{1,\dots,n+1\} \setminus S$, and we write $X^{(n+1)} \stackrel{\text{def}}{=} -\sum_{j=1}^{n} X^{(j)}$ and $Y^{(n+1)} \stackrel{\text{def}}{=} -\sum_{j=1}^{n} Y^{(j)}$.

Definition A.2. Let l be a prime number, and let G be a pro-l configuration space group [cf. [21, Definition 2.3, (i)]]. Write G^{ab} for the topological abelianization of G, i.e., the quotient of G by the closure of the commutator subgroup of G.

(i) It follows from [8, Theorem 1.6] that there exists a nonnegative integer n such that G contains a closed subgroup isomorphic to the direct product of n copies of $\mathbb{Z}_l$ but does not contain any closed subgroup isomorphic to the direct product of $n+1$ copies of $\mathbb{Z}_l$. We shall write

$$n(G)$$

for this [uniquely determined] nonnegative integer.

(ii) Let $N \subseteq G$ be a normal closed subgroup of G. Then we shall say that the normal closed subgroup N is *quasi-co-surface* [8, Definition 2.1, (iii)] if the quotient G/N is a pro-l surface group [i.e., a profinite group isomorphic to the maximal pro-l quotient of the étale fundamental group of a hyperbolic curve over a separably closed field of characteristic $\neq l$ — cf. [21, Definition 1.2]] that is not isomorphic to a free pro-l group of rank two.

(iii) We shall say that G is *of type* $(0,3)$ if there is no quasi-co-surface subgroup of G, and, moreover, the topological abelianization G^{ab} of G is not isomorphic to the direct product of $2n(G)$ copies of $\mathbb{Z}_l$.

(iv) We shall say that G is *of type* $(1,1)$ if there is no quasi-co-surface subgroup of G, and, moreover, the topological abelianization G^{ab} of G is isomorphic to the direct product of $2n(G)$ copies of $\mathbb{Z}_l$.

(v) Suppose that G does not have any quasi-co-surface subgroup, i.e., is either of type $(0,3)$ or of type $(1,1)$. Write (g,r) for the element of $\{(0,3),(1,1)\}$ such that G is of type (g,r). Let $V \subseteq G^{\mathrm{ab}} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ be a vector subspace of $G^{\mathrm{ab}} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ over $\mathbb{Q}_l$. Then we shall say that V is a *generalized fiber subspace of* $G^{\mathrm{ab}} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ *of co-length one* if there exist an isomorphism $\mathrm{Gr}(G) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \xrightarrow{\sim} \mathscr{L}_{(g,r,n(G))}$ [cf. Definition A.1, (i), (iii)], where we write $\mathrm{Gr}(G)$ for the graded Lie algebra over $\mathbb{Z}_l$ associated to the lower central series of G, of graded Lie algebras over $\mathbb{Q}_l$ and a subset S of $\{1,\ldots,n(G)+r\}$ of cardinality $4-2g$ such that the image of $V \subseteq G^{\mathrm{ab}} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ by the isomorphism $\mathrm{Gr}(G) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \xrightarrow{\sim} \mathscr{L}_{(g,r,n(G))}$ coincides with the subspace $W^S_{(g,r,n(G))} \subseteq \mathscr{L}_{(g,r,n(G))}$ [cf. Definition A.1, (ii), (iv)].

Definition A.3. Suppose that we are in the situation of Definition A.2. Let $N \subseteq G$ be a normal closed subgroup of G.

(i) We shall define a *generalized fiber subgroup of* G *of co-length zero* to be G.

(ii) Suppose that G has a quasi-co-surface subgroup of G. Then we shall say that N is a *generalized fiber subgroup of* G *of co-length one* if N is a minimal quasi-co-surface subgroup of G.

(iii) Suppose that G does not have any quasi-co-surface subgroup. Then we shall say that N is a *generalized fiber subgroup of* G *of co-length one* if G/N is elastic, and, moreover, the image of $N \subseteq G$ by the natural homomorphism $G \to G^{\mathrm{ab}} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ generates a generalized fiber subspace of $G^{\mathrm{ab}} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ of co-length one.

(iv) Observe that it follows from [8, Theorem 2,5, (i), (ii), (iii), (iv)] that every generalized fiber subgroup of co-length one of a pro-l configuration space group is a pro-l configuration space group. Let i be an element of $\{2,\ldots,n(G)\}$. Then we shall say that N is a *generalized fiber subgroup of* G *of co-length* i if there exists a sequence of closed subgroups $N = N_i \subseteq \ldots \subseteq N_1 \subseteq N_0 = G$ of G such that N_j is a generalized fiber subgroup of N_{j-1} of co-length one for each $j \in \{1,\ldots,i\}$.

Theorem A.4. *Let* l *be a prime number,* G *a pro-*l *configuration space group,* $m \leqslant n$ *positive integers,* k *a separably closed field of characteristic* $\neq l$, X^+ *a hyperbolic curve over* k, *and* I *a subset of* $\{1,\ldots,n+\varepsilon_{X^+}\}$ [*cf. Definition* 2.7, (i)]. *In particular, the hyperbolic curve* X^+ *is split* [*cf. Definition* 2.4]. *Let us fix a lifting* $\mathrm{Spec}(k) \to \mathscr{M}_{g,r}$ [*cf. Definition* 2.3, (ii)] *of the classifying morphism* $\mathrm{Spec}(k) \to \mathscr{M}_{g,[r]}$ [*cf. Definition* 2.3, (iii)] *of the hyperbolic curve* X^+ [*cf. Remark* 2.3.1]. *For each* $i \in \{0,\ldots,n\}$, *we shall write*

$$\Pi_i \stackrel{\mathrm{def}}{=} \pi_1^{\mathrm{tame}}(X^+_{(i)})^{\{l\}}$$

[*cf. Definition* 1.1, (iii)*; Definition* 2.6*; Lemma* 2.8, (i)*; Definition* 3.1, (i)]. *In particular, one verifies easily that the profinite group* Π_n *is a pro-l configuration space group. Let* $N \subseteq \Pi_n$ *be a normal closed subgroup of* Π_n*. Then the following two conditions are equivalent:*

(i) *The normal closed subgroup* $N \subseteq \Pi_n$ *of* Π_n *is a generalized fiber subgroup of* Π_n *of co-length* m [*cf. Definition* A.3, (iv)].

(ii) *There exists a subset* I *of* $\{1, \ldots, n + \varepsilon_{X^+}\}$ *of cardinality* $n - m$ *such that* N *coincides with the kernel of the continuous outer homomorphism* $\Pi_n \to \Pi_{n-\#I}$ *induced by the morphism* $\mathrm{pr}_I \colon X_{(n)} \to X_{(n-\#I)}$ [*cf. Definition* 2.7, (ii), (iii)].

Proof. This assertion is a formal consequence of [8, Theorem 2.5, (v)]. □

References

1 P. Deligne and D. Mumford, The irreducibility of the space of curves of given genus, *Inst. Hautes Études Sci. Publ. Math.* (1969), no. 36, 75–109.

2 E. M. Friedlander, The étale homotopy theory of a geometric fibration, *Manuscripta Math.* **10** (1973), 209–244.

3 M. Fried and M. Jarden, *Field arithmetic*, Third edition, Revised by Jarden, Ergeb. Math. Grenzgeb. (3), **11**, Springer-Verlag, Berlin, 2008.

4 Y. Hoshi, On the fundamental groups of log configuration schemes, *Math. J. Okayama Univ.* **51** (2009), 1–26.

5 Y. Hoshi, The exactness of the log homotopy sequence, *Hiroshima Math. J.* **39** (2009), no. 1, 61–121.

6 Y. Hoshi, Y. Hoshi, The Grothendieck conjecture for hyperbolic polycurves of lower dimension, *J. Math. Sci. Univ. Tokyo* **21** (2014), no. 2, 153–219.

7 Y. Hoshi, The absolute anabelian geometry of quasi-tripods, *Kyoto J. Math.* **62** (2022), no. 1, 179–224.

8 Y. Hoshi, A. Minamide, and S. Mochizuki, Group-theoreticity of numerical invariants and distinguished subgroups of configuration space groups, *Kodai Math. J.* **45** (2022), no. 3, 295–348.

9 Y. Hoshi and S. Mochizuki, Topics surrounding the combinatorial anabelian geometry of hyperbolic curves I: inertia groups and profinite Dehn twists, *Galois-Teichmüller theory and arithmetic geometry*, 659–811, *Adv. Stud. Pure Math.* **63**, *Mathematical Society of Japan, Tokyo*, 2012.

10 Y. Hoshi and S. Mochizuki, *Topics surrounding the combinatorial anabelian geometry of hyperbolic curves II: Tripods and combinatorial cuspidalization*, Lecture Notes in Mathematics, **2299**, Springer, Singapore, 2022.

11 Y. Hoshi and S. Mochizuki, Topics surrounding the combinatorial anabelian geometry of hyperbolic curves III: Tripods and Tempered fundamental groups, *Kyoto J. Math.* **65** (2025), no. 4, 787–875.

12 Y. Hoshi and S. Tsujimura, On the injectivity of the homomorphisms from the automorphism groups of fields to the outer automorphism groups of the absolute Galois groups, *Res. Number Theory* **9** (2023), no. 2, Paper No. 44, 13 pp.

13 K. Kato, Logarithmic structures of Fontaine-Illusie, *Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988)*, 191–224, *Johns Hopkins University Press, Baltimore, MD*, 1989.

14 F. F. Knudsen, The projectivity of the moduli space of stable curves, II: The stacks $M_{g,n}$, *Math. Scand.* **52** (1983), no. 2, 161–199.

15 A. Minamide, K. Sawada, and S. Tsujimura, On generalizations of anabelian group-theoretic properties, *Hiroshima Math. J.* **55** (2025), no. 2, 109–140.

16 S. Mochizuki, The local pro-p anabelian geometry of curves, *Invent. Math.* **138** (1999), no. 2, 319–423.

17 S. Mochizuki, A combinatorial version of the Grothendieck conjecture, *Tohoku Math. J.* **59** (2007), no. 3, 455–479.

18 S. Mochizuki, Absolute anabelian cuspidalizations of proper hyperbolic curves, *J. Math. Kyoto Univ.* **47** (2007), no. 3, 451–539.

19 S. Mochizuki, On the combinatorial cuspidalization of hyperbolic curves, *Osaka J. Math.* **47** (2010), no. 3, 651–715.

20 S. Mochizuki, Topics in absolute anabelian geometry I: Generalities, *J. Math. Sci. Univ. Tokyo* **19** (2012), no. 2, 139–242.

21 S. Mochizuki and A. Tamagawa, The algebraic and anabelian geometry of configuration spaces, *Hokkaido Math. J.* **37** (2008), no. 1, 75–131.

22 F. Pop, On Grothendieck's conjecture of birational anabelian geometry, *Ann. of Math. (2)* **139** (1994), no. 1, 145–182.

23 F. Pop, *On Grothendieck's conjecture of anabelian birational geometry II*, Heidelberg–Mannheim Preprint Reihe Arithmetik II, No. 16, Heidelberg 1995.

24 M. Saïdi and A. Tamagawa, A prime-to-p version of Grothendieck's anabelian conjecture for hyperbolic curves over finite fields of characteristic $p > 0$, *Publ. Res. Inst. Math. Sci.* **45** (2009), no. 1, 135–186.

25 M. Saïdi and A. Tamagawa, A refined version of Grothendieck's anabelian conjecture for hyperbolic curves over finite fields, *J. Algebraic Geom.* **27** (2018), no. 3, 383–448.

26 K. Sawada, Cohomology of the geometric fundamental group of hyperbolic polycurves, *J. Algebra* **508** (2018), 364–389.

27 K. Sawada, *Reconstruction of invariants of configuration spaces of hyperbolic curves from associated Lie algebras*, RIMS Preprint **1896** (November 2018).

28 J. Stix, Affine anabelian curves in positive characteristic, *Compositio Math.* **134** (2002), no. 1, 75–85.

29 J. Stix, Projective anabelian curves in positive characteristic and descent theory for log-étale covers, Dissertation, Rheinische Friedrich-Wilhelms-Universität Bonn, Bonn, 2002, Bonner Math. Schriften, **354**, Universität Bonn, Mathematisches Institut, Bonn, 2002.

30 A. Tamagawa, The Grothendieck conjecture for affine curves, *Compositio Math.* **109** (1997), no. 2, 135–194.

31 A. Tamagawa, On the tame fundamental groups of curves over algebraically closed fields of characteristic > 0, *Galois groups and fundamental groups*, 47–105, Math. Sci. Res. Inst. Publ., **41**, Cambridge Univ. Press, Cambridge, 2003.

32 S. Tsujimura, *Grothendick conjecture for hyperbolic curves over finitely generated fields of positive characteristic via compatibility of cyclotomes*, RIMS Preprint **1975** (July 2023). The latest version is available at `https://www.kurims.kyoto-u.ac.jp/~stsuji/`

33 I. Vidal, Contributions à la cohomologie étale des schémas et des log-schémas, Thèse, U. Paris-Sud (2001).

34 *Revêtements étales et groupe fondamental (SGA 1)*, Séminaire de géométrie algébrique du Bois Marie 1960–61, Directed by A. Grothendieck, With two papers by M. Raynaud, Updated and annotated reprint of the 1971 original, Doc. Math. (Paris), **3**, Société Mathématique de France, Paris, 2003.

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